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Cambridge Tracts in Mathematics
and Mathematical Physics

General Editors
J. G. LEATHEM, Sc.D.
G. H. HARDY, M.A., F.R.S.

No. 20

THE
ELEMENTARY DIFFERENTIAL GEOMETRY OF PLANE CURVES

BY

R. H. FOWLER, M.A.
Fellow of Trinity College, Cambridge

UNIVERSITY COLLEGE, NOTTINGHAM

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C

PREFACE

THIS tract is intended to present a precise account of the elementary differential properties of plane curves. The matter contained is in no sense new, but a suitable connected treatment in the English language has not been available.

As a result, a number of interesting misconceptions are current in English text books. It is sufficient to mention two somewhat striking examples. (*a*) According to the ordinary definition of an envelope, as the locus of the limits of points of intersection of neighbouring curves, a curve is not the envelope of its circles of curvature, for neighbouring circles of curvature do not intersect. (*b*) The definitions of an asymptote—(1) a straight line, the distance from which of a point on the curve tends to zero as the point tends to infinity; (2) the limit of a tangent to the curve, whose point of contact tends to infinity—are not equivalent. The curve may have an asymptote according to the former definition, and the tangent may exist at every point, but have no limit as its point of contact tends to infinity.

The subjects dealt with, and the general method of treatment, are similar to those of the usual chapters on geometry in any *Cours d'Analyse*, except that in general plane curves alone are considered. At the same time extensions to three dimensions are made in a somewhat arbitrary selection of places, where the extension is immediate, and forms a natural commentary on the two dimensional work, or presents special points of interest (Frenet's formulae). To make such extensions systematically would make the tract too long. The subject matter being wholly classical, no attempt has been made to give full references to sources of information; the reader however is referred at most stages to the analogous treatment of the subject in the *Cours* or *Traité d'Analyse* of de la Vallée Poussin, Goursat, Jordan or Picard, works to which the author is much indebted.

In general the functions, which define the curves under consideration, are (as usual) assumed to have as many continuous differential coefficients as may be mentioned. In places, however, more particularly at the beginning, this rule is deliberately departed from, and the greatest generality is sought for in the enunciation of any theorem. The determination of the *necessary and sufficient* conditions for the truth of any theorem is then the primary consideration. In the proofs of the elementary theorems, where this procedure is adopted, it is believed that this treatment will be found little more laborious than any rigorous treatment, and that it provides a connecting link between Analysis and more complicated geometrical theorems, in which insistence on the precise *necessary* conditions becomes tedious and out of place, and suitable *sufficient* conditions can always be tacitly assumed. At an earlier stage the more precise formulation of conditions may be regarded as (1) an important grounding for the student of Geometry, and (2) useful practice for the student of Analysis.

The introductory chapter is a collection of somewhat disconnected theorems which are required for reference. The reader can omit it, and to refer to it as it becomes necessary for the understanding of later chapters.

I wish to express my great indebtedness to the Editor, Mr G. H. Hardy, and also to Mr J. E. Littlewood and Dr T. J. I'A. Bromwich, for assistance and advice in the preparation of this tract.

R. H. F.

October 1919.

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CHAPTER I

INTRODUCTION

§ 1·10. We assume in this tract that the reader is acquainted with the ordinary elementary theorems of the differential and integral calculus, as developed, for example, in Hardy's *Pure Mathematics* (2nd Edition, 1914); we apply these theorems to the geometry of plane curves. We shall require more than is there given concerning implicit functions, especially algebraic functions and the curves defined thereby. Such theorems of this type as we require frequently are quoted with references in § 1·50. The more important special properties of algebraic functions are summarized in § 1·60.

We shall use freely the symbols $\sim$, O, o, whose use is now classical, and occasionally $\asymp$ and $\succcurlyeq$. The reader who is not acquainted with any of them will find the meaning of $\sim$, O, $\asymp$ and $\succcurlyeq$ explained in Hardy's tract 'Orders of Infinity' (*Cambridge Mathematical Tracts*, No. 12). The definition of o is as follows:

If $f(x)$ and $g(x)$ are any functions of x, and $g(x)$ is positive for all sufficiently large values of x, we write*

$$f(x) = o(g(x)),$$

when

$$|f(x)|/g(x) \to 0$$

as $x \to \infty$.

A similar definition applies when x tends to zero, or any other finite limit, instead of to infinity. The introduction of o into Analysis is due to Landau, *vide* Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. I, p. 59.

The symmetry of the differential notation and the use of direction cosines are of vital importance in three-dimensional geometry. They can be used with advantage in two dimensions and lend themselves at once to the necessary generalizations. They are therefore used freely here. It is, however, important that the reader should realise the

---

* Alternatively, it will be convenient for our purposes to allow $g(x)$ to be negative instead of positive in the above definition, and also in the definition of O. The only essential requisite in these definitions is that $g(x)$ should not vanish for large values of x.

precise nature of the statement made by a differential formula, and this is frequently emphasised.

A limited selection of examples is given at the ends of the chapters. Besides their more obvious function, these are intended to provide a summary of some of the more important extensions of the theorems proved in the text. References or sketches of a proof are therefore given in such cases, which should enable the reader to complete the proofs.

§ 1·20. **Plane curves.** We regard a plane curve as the locus of points satisfying the equations

$$x = \phi_1(t),$$
$$y = \phi_2(t),$$

for a given range of values of t ($t_0 \leqslant t \leqslant t_1$, say) for which $\phi_1(t)$, $\phi_2(t)$ are continuous single-valued functions of t. A point P on the curve is regarded as identified with a value of t. The variable t is real, and x and y are also always real. We consider throughout only real points and curves.

More information than this about $\phi_1(t)$ and $\phi_2(t)$ will always be required, the amount varying from problem to problem. We may specify conditions to be satisfied by $\phi_1(t)$ and $\phi_2(t)$ either

(1) at a point P, *i.e.* when $t = t_0$,

or (2) in the neighbourhood of or "near" a point P, *i.e.* in the neighbourhood of t_0,

or (3) throughout the interval PQ, *i.e.* when $t_0 \leqslant t \leqslant t_1$.

We say that the point $P(t)$ lies between the points $Q_1(t_1)$ and $Q_2(t_2)$ on the curve, when $t_1 < t < t_2$; also that the point $Q(t)$ tends to $P(t_0)$, or $Q \rightarrow P$, P and Q being points on the curve, when $t \rightarrow t_0$.

A particular case of great importance occurs when $x = t$ or $y = t$, and the curve is given in one of the forms

$$y = \phi_2(x), \quad x = \phi_1(y).$$

Curves may also be defined by implicit functions. We return to these in § 1·50.

We shall frequently be concerned with straight lines, circles and other curves which depend on certain parameters, and shall study their behaviour as the parameters tend to certain limiting values. In general, suppose the curve is defined by the equations

$$(1·21) \qquad x = \phi_1(t, \alpha, \beta, \ldots), \quad y = \phi_2(t, \alpha, \beta, \ldots),$$

or by the equation

$$(1·22) \qquad f(x, y, \alpha, \beta, \ldots) = 0.$$

Let $\alpha \to \alpha_0$, $\beta \to \beta_0$, ... and suppose that

$$\phi_1(t, \alpha, \beta, \dots) \to \chi_1(t),$$
$$\phi_2(t, \alpha, \beta, \dots) \to \chi_2(t),$$
$$f(x, y, \alpha, \beta, \dots) \to g(x, y).$$

Then we shall say that *the curve* $x = \chi_1(t)$, $y = \chi_2(t)$ *is the limit of the curve defined by* 1·21 *and that the curve* $g(x, y) = 0$ *is the limit of the curve defined by* 1·22.

For example we shall define the tangent to any curve at a point P as the limit of the chord PQ as $Q \to P$, the word limit being interpreted in the above sense. The chord may be

$$y = m(\xi) x + c(\xi),$$

where ξ is the parameter of Q. The limit of this chord is

$$y = px + d,$$

where $m(\xi) \to p$, $c(\xi) \to d$ as $\xi \to \xi_0$. Thus, in the case of an algebraic curve such as the above straight line, whose coefficients depend on a parameter or parameters, we regard as the limit of the given curve that curve whose coefficients are the limits of the coefficients of the given curve.

We shall often go further than this and regard the curves defined by 1·21 or 1·22 as approximate representations of their limiting curves. It is then important to be able to describe shortly the closeness or order of such an approximation. Suppose for example that $\alpha = \alpha_0 + \delta\alpha$, and that for any given values of x and y

$$f(x, y, \alpha) = g(x, y) + O(\delta\alpha)^q,$$

as $\delta\alpha \to 0$; in this relation q is a positive integer independent of x and y, while the constant implied by the O may (and in general will) depend essentially on the choice of x and y. Then we shall say that *the curve* $f(x, y, \alpha) = 0$, *when* α *is near* α_0, *represents the limiting curve* $g(x, y) = 0$, *with an error* $O(\delta\alpha)^q$. When $f(x, y, \alpha)$ is a polynomial in x and y with one non-zero coefficient independent of α, this statement is equivalent to saying that all the coefficients of x and y in $f(x, y, \alpha)$ differ from their limiting values by terms of order $(\delta\alpha)^q$.

§ 1·30. We frequently attempt to impose the minimum conditions that enable us to make a definition or to prove a theorem. In considering the properties of the curve $y = f(x)$ at any point $P(x_0)$ on the curve, we shall always impose the condition that $f'(x_0)$ exists*. This is of course the necessary and sufficient condition for the existence of a tangent not parallel to the axis of y. We then proceed to consider the properties of the curve in the neighbourhood of P, and for this purpose assume in general that $f'(x_0)$ is continuous at P. In this case the curve is always *rectifiable*—it is always possible to assign a meaning to the *length* of an arc of the curve in this neighbourhood. We therefore ignore the question of the necessary and sufficient conditions that a curve should be rectifiable*; a discussion of this question would be out of place here.

---

* See § 2·10, note.

Further assumptions are then introduced, such as the existence or continuity of $f''(x_0)$, etc., as required by the problem discussed.

We may mention in passing that the assumption of the existence of $f'(x_0)$ implies rather more than is at once apparent from the definition of a differential coefficient, and that the additional implications are of some geometrical interest (see Note A).

§ 1·40. **Choice of axes. Invariant relations.** We shall assume in general that the axes of coordinates to which our curves are referred are rectangular. It will usually be sufficiently clear to the reader when a theorem remains unaltered by permitting the use of oblique axes.

It is often convenient, in the proof of some general property of a curve, to simplify the proof by referring the curve to a special set of axes, such as the tangent and normal at a point on the curve. It is therefore important to be able to assert that a property proved with a special set of axes is true of curves in general, *i.e.* whatever the axes of reference. *It is permissible to make this assertion owing to the invariance of the formal expressions of lengths and angles for the most general changes of rectangular axes.*

The restriction to rectangular axes is, of course, unnecessary, but a consideration of this case is sufficient for the argument. The general rectangular transformation is

$$x = x' \cos\theta - y' \sin\theta + a,$$

$$y = x' \sin\theta + y' \cos\theta + b,$$

where a, b, θ are any constants. It is easily verified that

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2.$$

This is the property of invariance of length. For the invariance of the (tangent of the) angle between two straight lines, it is easy to see that if the lines are

$$y = m_1 x + c_1, \quad y = m_2 x + c_2,$$

---

* See § 2·50, note.

and they transform into

$$y' = m_1'x' + c_1', \quad y' = m_2'x' + c_2',$$

then

$$\frac{m_1' - m_2'}{1 + m_1'm_2'} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Thus all the ordinary metrical properties of curves, which depend on the relations between points on the curves themselves and their tangents and normals, may be established for any system of axes and asserted to hold true in general.

For example, the curvature of a curve is defined in the usual way as

$$\underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta\psi}{\delta s},$$

where δs is the length of an arc of the curve, $\delta\psi$ the angle between the tangents at the ends of the arc, and δs and $\delta\psi$ are invariant for any change of axes. Hence the above limit, if it exists when the curve is referred to one system of axes, will exist and be equal for all others. It is shown in the usual way that the value of this limit is

$$\frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$$

This expression is therefore an invariant for the general change of rectangular axes, as may be directly verified. When we wish to prove properties of the centre of curvature of a point on a curve, we naturally refer the curve to the tangent and normal at the point considered. The value of the above invariant is then $(d^2y/dx^2)_0$, or $f''(0)$, and the algebra is greatly simplified.

§ 1·50. **Implicit functions.** In addition to the forms of § 1·20, curves may also be defined by implicit functional relations between the coordinates x and y of the type

$$f(x, y) = 0.$$

This case is reduced to the explicit form by the fundamental existence theorem for implicit functions*. We quote here the form most useful for our purposes.

THEOREM 1·51. **Existence theorem for implicit functions.**

Suppose that $F(x, y)$ *is a function of the two real variables* (x, y) *satisfying the conditions:*

(1) *it is real, one-valued, and continuous, and possesses a continuous partial differential coefficient* F_y' *in the neighbourhood of* (x_0, y_0):

---

* Hardy, *Pure Mathematics*, 2nd Ed., p. 192; Goursat, *Cours d'Analyse Mathématique*, 2nd Ed., Vol. I, Chap. III. The former will in future be referred to as Hardy, *P M.*, and the latter as Goursat, for shortness.

(2) $$F(x_0, y_0) = 0, \quad F_y'(x_0, y_0) \neq 0.$$

Then, (a) there exists a unique function $y = \phi(x)$ *which, when substituted in the equation* $F(x, y) = 0$, *satisfies it identically for all values of* x *in the neighbourhood of* x_0*:*

(b) $\phi(x)$ *is real and continuous in this neighbourhood, and* $\phi(x) \rightarrow y_0$ *as* $x \rightarrow x_0$.

If further F_x' *exists and is continuous in the neighbourhood of* (x_0, y_0), *the function* $y = \phi(x)$ *possesses a continuous differential coefficient in this neighbourhood, and*

(1·511) $$F_x' + F_y'(dy/dx) = 0,$$

so that

$$dy/dx = \phi'(x) = -F_x'/F_y'.$$

If further all the n^{th} *partial differential coefficients of* $F(x, y)$ *exist and are continuous in the neighbourhood of* (x_0, y_0), $d^n y/dx^n$ *exists and is continuous in this neighbourhood, and may be calculated by the usual rules.*

It may be noted that the extensions of the main existence theorem as quoted assume more than is required about F_x'. In order that $\phi(x)$ may have a differential coefficient at x_0, satisfying equation 1·511, the necessary and sufficient extra condition, over and above the conditions of the main theorem, is simply that F_x' exists at (x_0, y_0). This can be proved by an easy revision of the argument given by Goursat.

More general existence theorems applying to n functions of m independent variables are sometimes required, for example in the theory of contact of curves and surfaces. In such cases the reader is referred to Goursat*.

Cases of exception. If $F_y'(x_0, y_0) = 0$ the theorem breaks down, but if F_x' exists and is continuous, and $F_x'(x_0, y_0) \neq 0$, we can still apply the theorem with the roles of x and y interchanged, and obtain a unique real solution in the form $x = \phi(y)$. It is only if

$$F_y'(x_0, y_0) = F_x'(x_0, y_0) = 0,$$

that the breakdown of the theorem is complete. In this case (x_0, y_0) is called a **singular point.** We shall assume that there are only a finite number of such points in any region with which we deal. The question of the existence of solutions in the neighbourhood of such a point is discussed by Goursat, p. 102*. We shall return to these points and their geometrical properties in Chapter VI.

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* See also de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, Vol. I, p. 169. This book will be referred to in future as d.l.V.P. for shortness. References will be given to the 3rd Ed. of Vol. I, and the 2nd Ed. of Vol. II.

A particular choice of axes, referred to which the equation of a curve takes a particularly simple form, is frequently desirable. The validity of the necessary change of axes to convert a general curve into the required particular form may be established as follows.

Let us call for the moment an ordinary point on a curve

$$\text{(i)}\quad y=f(x),$$

a point at which $f'(x)$ exists and is continuous; on a curve

$$\text{(ii)}\quad x=\phi_1(t),\ y=\phi_2(t),$$

a point at which $\phi_1'(t)$, $\phi_2'(t)$ exist and are continuous and not both zero; and on a curve

$$\text{(iii)}\quad F(x, y)=0,$$

a point at which F_x', F_y' exist and are continuous and not both zero.

We can then change the origin to the point near which we require the form of the curve, and rotate the axes until in the new system of coordinates (ξ, η)

$$\text{(i)}\ (d\eta/d\xi)_0=0,\ \textit{or}\ \text{(ii)}\ (d\eta/dt)_0=0,\ \textit{or}\ \text{(iii)}\ (F_\xi')_0=0,$$

and also $$\text{(ii)}\ (d\xi/dt)_0\neq 0,\ \textit{or}\ \text{(iii)}\ (F_\eta')_0\neq 0.$$

We thus arrive at the following theorem.

THEOREM 1·52. *By the choice of suitable axes a curve, in the neighbourhood of an ordinary point (as defined above), can always be expressed in the form*

$$y=f(x),\ (f(0)=f'(0)=0).$$

After what precedes this is obvious in case (i) and follows at once from 1·51 in case (iii). In case (ii) 1·51 may be applied to establish the existence of a function ψ such that

$$t=\psi(\xi),$$

where $\psi(0)=t_0$, $\psi'(0)\neq 0$, and $\psi'(\xi)$ is continuous near $\xi=0$. Theorem 1·52 then follows as before.

The assumptions as to the nature of an ordinary point are more than are necessary for the truth of Theorem 1·52, but we shall only require the theorem in cases in which these assumptions are required for other reasons.

§ 1·60. **Algebraic curves.** If the function $F(x, y)$ is a polynomial in x and y, then y is said to be an algebraic function of x, or the equation $F(x, y)=0$ is said to define an algebraic curve. This is the most important case of an implicit function. The relevant special properties of such curves may be roughly summarized as follows.

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* See also d.l.V.P., Vol. II, Chap. IX.

(1) Every special assumption that we require to make as to the nature of our curve is always true if the curve is algebraic, with the possible exception of a finite number of points.

(2) In particular there are at most a finite number of points at which

$$F(x, y) = F_x'(x, y) = F_y'(x, y) = 0,$$

near which the form of the curve is not determined by Theorem 1·51. In the neighbourhood of any such point the form of the curve can always be determined (see Chapters VI and VII) by a suitable extension of this theorem.

(3) The study of the form of the curve near infinity, *i.e.* as $x \to \infty$ or $y \to \infty$ or both, can always be reduced by a suitable substitution to the study of a similar curve in the neighbourhood of a finite point.

In making the foregoing statements we have tacitly assumed that $F(x, y)$ is not, for instance, of the form $\{G(x, y)\}^2$, where G is a polynomial, for in such a case every point of the curve is a singular point. Such cases of exception are trivial from the present point of view and may be guarded against by restricting ourselves to polynomials which are *irreducible*, that is to say, without factors that are themselves polynomials.

CHAPTER II

THE ELEMENTARY PROPERTIES OF TANGENTS AND NORMALS

§ 2·10. Definition and existence of the tangent.

DEFINITION. *The* **tangent** *to a curve at the point P is the limit (if it exists) of the straight line PQ, when $Q \to P$ along the curve.*

THEOREM 2·11. *The necessary and sufficient condition that the curve $y = f(x)$ should have a tangent at $P(x_0, f(x_0))$, not parallel to the axis of y, is that $f'(x_0)$ should exist. The equation of the tangent is then*

$$y - f(x_0) = f'(x_0)(x - x_0)^*.$$

---

* If we admit infinite differential coefficients, and agree that $f'(x_0)$ has the value $+\infty$ (or $-\infty$) if

$$\underset{x \to x_0}{\mathrm{Lt}} \frac{f(x) - f(x_0)}{x - x_0} = +\infty \text{ (or } -\infty\text{)},$$

the phrase "not parallel to the axis of y" may be omitted and the words "if $f'(x_0)$ is finite, or $x - x_0 = 0$ if $f'(x_0) = \pm\infty$" inserted at the end of the theorem.

The question is entirely one of phraseology, though perhaps more caution is required if infinite limits are admitted. We shall nowhere admit them in this tract.

(1) The condition is necessary. For if such a tangent exists, the line PQ has a limit of the form $y = Ax + B$. Now, if Q is the point $(\xi, f(\xi))$, the equation of PQ is

$$y - f(x_0) = \frac{f(\xi) - f(x_0)}{\xi - x_0}(x - x_0),$$

and therefore

$$\operatorname*{Lt}_{\xi \to x_0} \frac{f(\xi) - f(x_0)}{\xi - x_0}$$

exists, *i.e.* $f'(x_0)$ exists.

(2) The condition is sufficient. For if $f'(x_0)$ exists, then

$$\frac{f(\xi) - f(x_0)}{\xi - x_0}$$

tends to a finite limit as $\xi \to x_0$, and therefore PQ has a limit, not parallel to the axis of y, which is the tangent at P.

THEOREM 2·12. *In order that the curve*

$$x = \phi_1(t), \quad y = \phi_2(t)$$

may have a tangent at the point (t_0), *it is sufficient that* $\phi_1'(t_0)$ *and* $\phi_2'(t_0)$ *should both exist and not both be zero.*

The equation of the tangent is then

(2·121) $$(y - \phi_2(t_0))\,\phi_1'(t_0) = (x - \phi_1(t_0))\,\phi_2'(t_0)^*.$$

The equation of the chord PQ may be written

$$\frac{\phi_1(t) - \phi_1(t_0)}{t - t_0}(y - \phi_2(t_0)) = \frac{\phi_2(t) - \phi_2(t_0)}{t - t_0}(x - \phi_1(t_0)),$$

which has the required limit as $t \to t_0$, under the stated conditions.

THEOREM 2·13. *In order that the curve*

$$f(x, y) = 0$$

may have a tangent at (x_0, y_0), *it is sufficient that* f_x' *and* f_y' *should be continuous in the neighbourhood of* (x_0, y_0) *and not both be zero at* (x_0, y_0).

The equation of the tangent is then

(2·131) $$(x - x_0)f_x'(x_0, y_0) + (y - y_0)f_y'(x_0, y_0) = 0.$$

---

* A more useful form in practice is

(2·122) $$\frac{y - \phi_2(t_0)}{\phi_2'(t_0)} = \frac{x - \phi_1(t_0)}{\phi_1'(t_0)},$$

or more shortly

(2·123) $$\frac{y - y_0}{y_0'} = \frac{x - x_0}{x_0'}.$$

If one of x_0' or y_0' is zero, the equation must of course be interpreted to mean that the corresponding numerator vanishes.

This is a direct consequence of the Existence Theorem 1·51 and Theorem 2·11. If for example $f_y'(x_0, y_0) \neq 0$, the mere existence of $f_x'(x_0, y_0)$ is sufficient.

§ 2·20. **The tangent as the limit of the chord.** The tangent may be the limit of a chord of more general type than that used in the definition of § 2·10. This is shown by the following theorem.

THEOREM 2·21. *If $f'(x)$ is continuous at x_0, the tangent at x_0 to the curve $y=f(x)$ is the limit of the straight line Q_1Q_2 when $Q_1, Q_2 \rightarrow P$ along the curve.*

It is not sufficient here that $f'(x)$ should exist. Something more is required and continuity is a simple and sufficient condition. *If however $Q_1, Q_2 \rightarrow P$ from opposite sides*, then the chord Q_1Q_2 tends to the tangent at P provided only that the tangent exists. See Note A.

The equation of the chord Q_1Q_2 is

$$y-f(x_1)=\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1).$$

The hypothesis of the continuity of $f'(x)$ at x_0 implies the existence of $f'(x)$ at all neighbouring points to P. Hence Q_1 and Q_2 may be taken near enough to P for $f'(x)$ to exist at all points of the interval $x_1 \leqslant x \leqslant x_2$. Therefore by the mean value theorem

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(x_1+\theta(x_2-x_1)), \quad (0<\theta<1).$$

But since $f'(x)$ is continuous at x_0, $f'(x_1+\theta(x_2-x_1)) \rightarrow f'(x_0)$, when $Q_1, Q_2 \rightarrow P$, and the limit of the chord is

$$y-f(x_0)=f'(x_0)(x-x_0),$$

i.e. the tangent at P.

Other forms of the equation of the curve may be treated in a similar manner.

We shall speak of a curve as having *a continuous tangent at P*, when, as $Q \rightarrow P$, the tangent at Q tends to the tangent at P. It is easily seen that the necessary and sufficient condition for this in the case of $y=f(x)$ is that $f'(x)$ should be continuous at P, and, for the other two forms, that sufficient conditions are that $\phi_1'(t)$ and $\phi_2'(t)$ should be continuous, or that f_x' and f_y' should be continuous, respectively, at P.

§ 2·30. **Definition and equations of the normal.** DEFINITION. *The* **normal** *to a curve at the point P is a straight line through P at right angles to the tangent.*

The equations of the normal in the three cases studied above are respectively (the axes of coordinates being as always rectangular)

$$f'(x_0)(y-f(x_0))+(x-x_0)=0, \tag{2·31}$$

$$(y-\phi_2(t_0))\,\phi_2'(t_0)+(x-\phi_1(t_0))\,\phi_1'(t_0)=0, \tag{2·32}$$

$$\frac{x-x_0}{f_x'(x_0,y_0)}=\frac{y-y_0}{f_y'(x_0,y_0)}\text{*}. \tag{2·33}$$

§ 2·40. **The geometrical meaning of differentials.** Consider the curve

$$y=f(x)$$

with a tangent at P, the point (ξ, η), not parallel to the axis of y.

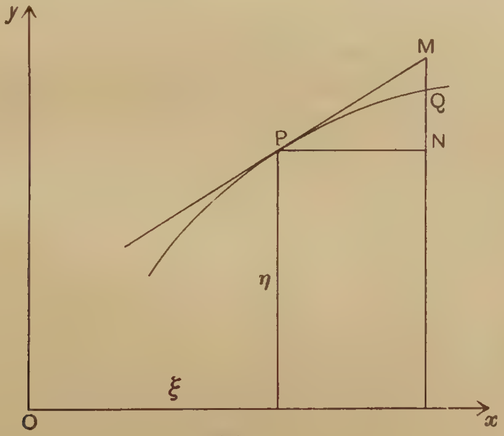

Fig. 1.

Let Q be a neighbouring point of abscissa $\xi+\delta x$, M the point on the ordinate at Q where it cuts the tangent at P, and PN a line through P parallel to Ox, the axis of x. Then

$$QN=\delta y,\ PN=\delta x.$$

Also the equation of the tangent at P is

$$y-f(\xi)=f'(\xi)(x-\xi),$$

so that

$$MN=f'(\xi)\,\delta x=df(x),$$

and therefore

$$MN=dy. \tag{2·41}$$

---

* This must be interpreted as in § 2·10 when either $f_x'(x_0,y_0)$ or $f_y'(x_0,y_0)$ vanishes.

Thus *the differential of y is the increase of the ordinate of a point on the tangent corresponding to an increase of abscissa δx.* This equality holds equally with oblique axes. Of course δx may be an increase of either sign, the increase of the ordinate being reckoned algebraically.

§ 2·50. **Arcs and their differentials.** Any arc of a plane curve which has a continuous tangent is *rectifiable**, and the length of the arc measured from a point P to a point Q (the coordinates being rectangular cartesians) is given by

$$s = \int_{x_0}^{x} \{1 + [f'(x)]^2\}^{\frac{1}{2}}\, dx, \tag{2·51}$$

$$s = \int_{t_0}^{t} \{[\phi_1'(t)]^2 + [\phi_2'(t)]^2\}^{\frac{1}{2}}\, dt, \tag{2·52}$$

for curves of the corresponding forms. To avoid ambiguities of sign it has been necessary to assign the direction in which s is to be regarded as increasing along the curve, when measured from a fixed point P. It it usually convenient, though not essential (see § 2·60), to take this direction as the direction of x increasing or t increasing, as has been done in 2·51 and 2·52 above. Taking 2·52 and using differentials we obtain†

$$ds = \{[\phi_1'(t)]^2 + [\phi_2'(t)]^2\}^{\frac{1}{2}}\, dt, \tag{2·53}$$

$$(ds)^2 = (dx)^2 + (dy)^2 ‡. \tag{2·531}$$

§ 2·60. **Conventions of sign.** We must now make certain conventions to avoid the repeated occurrences of ambiguities of sign. We have three positive directions to assign, namely, the positive axis of x,

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* See d.l.V.P., Vol. I, pp. 303, 368. For the study of rectifiable curves, more particularly of the *necessary* and sufficient conditions for rectifiability, see d.l.V.P., Vol. I, p. 380; Jordan, Vol. I, p. 99. For the properties of continuous curves in general, see d.l.V.P., Vol. I, p. 374; Jordan, Vol. I, p. 90. The book referred to under the latter title is Jordan's *Cours d'Analyse*, 3rd Ed.

† By the usual rule for differentiating an integral with a *continuous* integrand with respect to the upper limit. The assumption of a continuous tangent is made throughout this section.

‡ Note that 2·531 or its equivalent $ds = \{1 + (dy/dx)^2\}^{\frac{1}{2}} dx$ *is not the source of* 2·52 *but a deduction therefrom.* Equation 2·51 or 2·52 is fundamental, for it is a direct deduction from the definition of the length of an arc as the limit of an inscribed polygon, and until 2·51 has been established no meaning can be attached to 2·531.

The reader will of course bear in mind that 2·531 is *not* the same thing as the equation $(\delta s)^2 = (\delta x)^2 + (\delta y)^2$, which is false except for a straight line: 2·531 expresses in the differential notation the fact that, as $\delta x, \delta y \longrightarrow 0$,

$$(\delta s)^2 \sim (\delta x)^2 + (\delta y)^2.$$

i.e. the direction of x increasing, the positive axis of y, and the direction, clockwise or counter-clockwise, in which an angle is to be reckoned positive when measured from the positive axis of x. Any two of these three positive directions can be arbitrarily chosen without introducing ambiguities into any of our formulae, but when this has been done, the third positive direction cannot be so chosen if ambiguities are to be avoided. We shall therefore make the following convention.

(A) *The positive axis of y makes an angle $+\frac{1}{2}\pi$ with the positive axis of x.*

With this convention all our formulae remain correct whatever choice is made of the directions in which two of the quantities x, y, and the angle are reckoned positive. We shall, however, in general, suppose that all angles are reckoned positive in a counter-clockwise direction from the positive axis of x. This is convenient though unnecessary. When a concise name is required, we shall denote the positive axes of x and y by Ox and Oy, and similarly the negative axes by Ox' and Oy'.

We have now to assign positive directions along the tangent and normal at any point of a curve, and the direction of s increasing along the curve. We can assign arbitrarily the direction of s increasing along the curve, but once this has been done, no further liberty of choice remains if ambiguities are to be avoided. We make the following conventions.

(B) *The positive direction of the tangent is the direction of a line drawn along the tangent in the direction of s increasing.*

This direction will be spoken of as the direction of the tangent or simply as "the tangent" when no misunderstanding can arise.

(C) *The positive direction of the normal makes an angle $+\frac{1}{2}\pi$ with "the tangent".*

This may be spoken of as the direction of the normal or simply as "the normal". If "the tangent" is the same as Ox, then "the normal"[*] is the same as Oy.

§ 2·610. **Further differential formulae.** If ψ[†] is the angle made by the tangent with Ox, we have, in all cases, with the above conventions,

$$\tan\psi = dy/dx = f'(x).$$

It follows at once that

$$\sin\psi = \pm\, dy/ds, \quad \cos\psi = \pm\, dx/ds,$$

---

[*] We shall not in future emphasise this meaning by inverted commas.

[†] This use of ψ is constant throughout the rest of Chapters II and III.

and both signs must always be positive[*]. Therefore

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi, \tag{2·611}$$

$$dx = \cos\psi\, ds, \quad dy = \sin\psi\, ds\,[\dagger]. \tag{2·612}$$

If l, m[‡] are the direction cosines of the tangent, then

$$l = \cos\psi, \quad m = \sin\psi,$$

$$\frac{dx}{ds} = l, \quad \frac{dy}{ds} = m, \tag{2·613}$$

$$dx = l\,ds, \quad dy = m\,ds\,[\dagger], \tag{2·614}$$

$$ds = l\,dx + m\,dy\,[\dagger]. \tag{2·615}$$

All these forms express the same facts in different notations, all of utility. The figures illustrate the last sections. The positive directions of tangent and normal are denoted by Pt and Pn respectively. The two curves shown, aa and bb, have the same tangent and normal at P. It must always be remembered that none of the formulae of this section

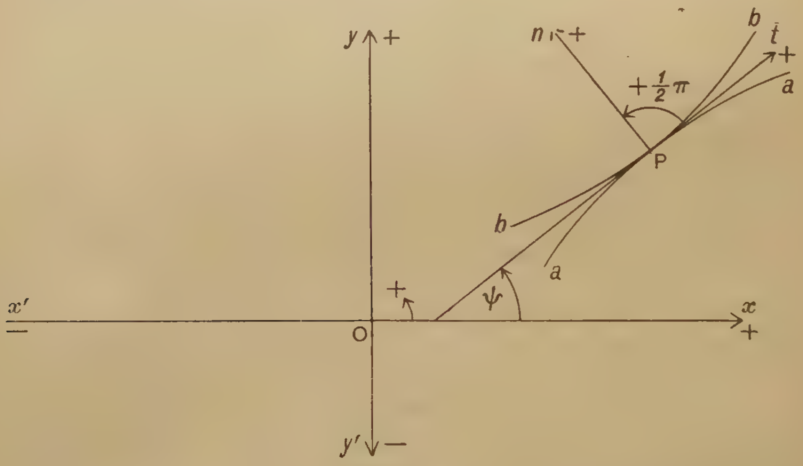

Fig. 2 (*a*).

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[*] The conventions were of course chosen so that this should be so.

[†] The reader will bear in mind that these differential relations are exact and are not the same as the equations

$$\delta x = \cos\psi\, \delta s, \quad \delta y = \sin\psi\, \delta s,$$

etc. These latter are in fact false unless the curve is a straight line. See equation 2·531, note.

[‡] This use of l, m is constant throughout the rest of Chapters II and III.

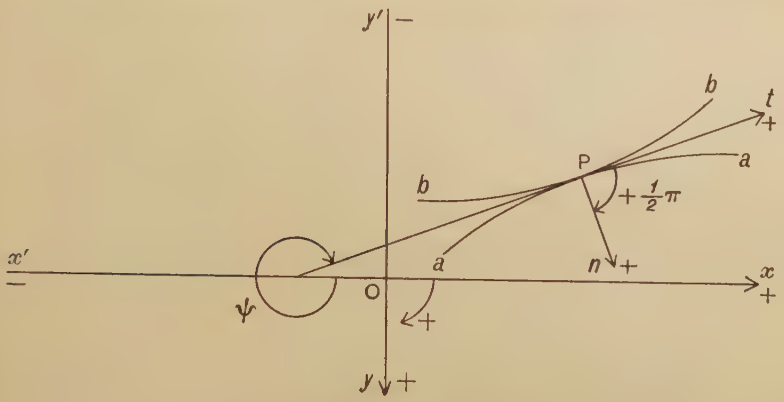

Fig. 2 (*b*).

are affected by the choice of the direction of s increasing. We shall return to these formulae when treating of curvature.

§ 2·620. **Limiting ratios of arcs, chords, and tangents.** The following theorems deal with the limiting ratios of arcs, chords and tangents.

THEOREM 2·621. *The ratio of the arc and the corresponding chord tends to* 1 *as the arc tends to* 0.

For
$$(\delta s)^2 = (ds)^2 (1 + o(1)),$$
$$(\delta x)^2 + (\delta y)^2 = (dx)^2 (1 + o(1)) + (dy)^2 (1 + o(1)),$$
$$= \{(dx)^2 + (dy)^2\} (1 + o(1)),$$
and therefore
$$(\delta s)^2 \sim (\delta x)^2 + (\delta y)^2,$$
which latter is the square of the chord.

It is easy to show more exactly that, if $f(x)$ has a continuous third differential coefficient (with similar conditions for the other forms of curve),

$$\text{Arc} - \text{Chord} = O(\delta s)^3. \tag{2·622}$$

As a corollary of this it is easy to prove that if μ is the greatest distance between the arc and the chord then

$$\mu = O(\delta s)^2. \tag{2·623}$$

One further step shows that these lengths are effectively of these orders, except for isolated points, unless the curve is a straight line[*]. These results are frequently useful.

The following result, also useful, is of the same nature as the foregoing. We omit the proof[†].

---

[*] See Ex. I, 1–3.

[†] See Ex. I, 5.

THEOREM 2·624. *If PT, QT, are tangents to the curve $y=f(x)$ at P and Q, intersecting at T, then*

$$PT+QT=\delta s+O(\delta s)^3, \tag{2·625}$$

as $Q\to P$; and, if also $f''(x)\neq 0$ at P,

$$PT-QT=O(\delta s)^2\text{*}. \tag{2·626}$$

We assume a continuous third differential coefficient. Equations 2·625 and 2·626 hold of course for any form of curve, with a suitable condition to replace $f''(x)\neq 0$. This condition may be stated in the general form that P *must not be a point at which the tangent is stationary*†.

§ 2·70. **Tangents in polar coordinates.** We shall content ourselves with considering a curve of the form $r=f(\theta)$, or one that can be put into that form in the neighbourhood of (r_0, θ_0), the point under consideration.

THEOREM 2·71. *The necessary and sufficient condition that the curve $r=f(\theta)$ may have a tangent at (r_0, θ_0), which is not the radius vector to that point*‡, *is that $f'(\theta_0)$ exists.*

The equation of the tangent is then

$$\frac{f(\theta_0)}{r}=\cos(\theta-\theta_0)-\frac{f'(\theta_0)}{f(\theta_0)}\sin(\theta-\theta_0). \tag{2·711}$$

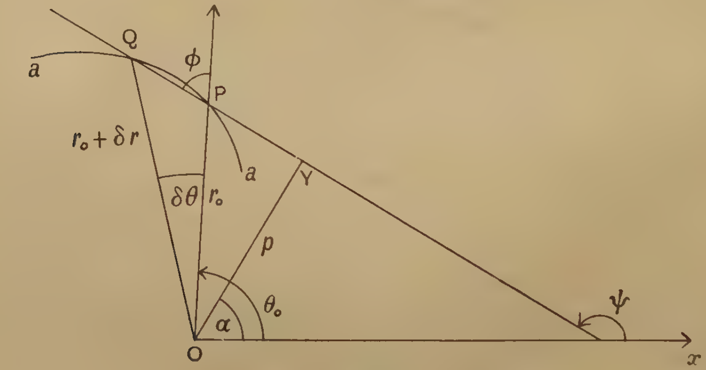

Fig. 3.

If the line QP has a limit as $Q\to P$ which is not the line OP, ϕ (see Figure 3) has a limit which is not 0 or π and so $\sin\phi$ has a non-zero limit. But

$$(r_0+\delta r)\sin(\phi-\delta\theta)-r_0\sin\phi=0,$$

---

* For the case $f''(x)=0$ see Ex. II, 4.

† See § 3·10 and Ex. II, 4.

‡ See Theorem 2·11, footnote.

and therefore as $Q \to P$, *i.e.* as δr and $\delta\theta$ tend to zero,

$$\frac{\delta r}{\delta\theta} = (r_0 + \delta r)\left(\frac{\sin\delta\theta}{\delta\theta}\right)\left(\frac{\cos\phi}{\sin\phi}\right)\left(\frac{1}{\cos\delta\theta}\right) + \frac{r_0(1-\cos\delta\theta)}{\delta\theta\cos\delta\theta}.$$

The right-hand side tends to a limit, since each term has a limit, and therefore $\delta r/\delta\theta$ has a limit; *i.e.* $f'(\theta_0)$ exists. Conversely, the above reasoning may be reversed, and the theorem is proved.

To find the equation of the tangent when $f'(\theta)$ exists, we proceed as follows. Let (p, α) be the polar coordinates of the foot of the perpendicular from O on PQ. Then p and α both tend to finite limits when $Q \to P$. The equation of QPY is

$$r\cos(\theta-\alpha)=p; \tag{2·712}$$

and, since P and Q lie on this line,

$$r_0\cos(\theta_0-\alpha)=p,$$

$$(r_0+\delta r)\cos(\theta_0+\delta\theta-\alpha)=p.$$

Therefore, in the limit, p and α referring to the tangent,

$$f(\theta_0)\cos(\theta_0-\alpha)=p, \tag{2·713}$$

$$f'(\theta_0)\cos(\theta_0-\alpha)-f(\theta_0)\sin(\theta_0-\alpha)=0. \tag{2·714}$$

To obtain the tangent we find, from 2·712 and 2·713,

$$\frac{f(\theta_0)}{r}=\frac{\cos(\theta-\alpha)}{\cos(\theta_0-\alpha)}=\frac{\cos(\theta_0-\alpha+\theta-\theta_0)}{\cos(\theta_0-\alpha)}$$

$$=\cos(\theta-\theta_0)-\tan(\theta_0-\alpha)\sin(\theta-\theta_0),$$

and so, by 2·714,

$$\frac{f(\theta_0)}{r}=\cos(\theta-\theta_0)-\frac{f'(\theta_0)}{f(\theta_0)}\sin(\theta-\theta_0). \tag{2·72}$$

§ 2·730. **Conventions of sign and differential formulae in polar coordinates.** We make the following conventions.

The initial line of θ is Ox and θ is measured positive counter-clockwise. When the tangent is continuous, so that the curve is rectifiable, s is so chosen as to increase with θ. The angle ϕ between the tangent and radius vector is reckoned positive counter-clockwise from the positive radius vector* to the positive direction of the tangent. We then have in all cases

$$\psi \equiv \theta+\phi, \text{ (Mod. } 2\pi). \tag{2·731}$$

---

[*] The positive radius vector is to mean the direction of an arm drawn from O making an angle $+\theta$ with the initial line Ox. It must be remembered that negative values of r have to be allowed for.

The figures 4 (*a*) and 4 (*b*) illustrate this equation.

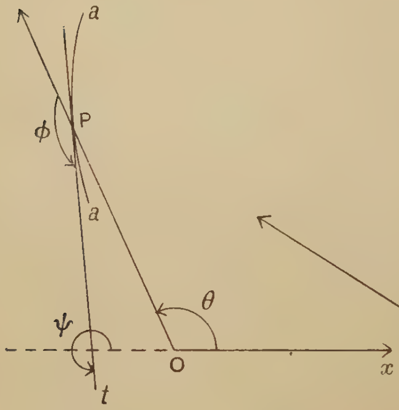

Fig. 4 (*a*).

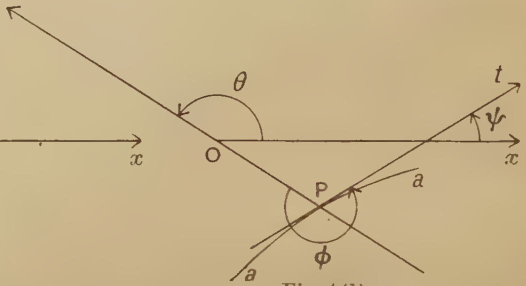

Fig. 4 (*b*).

Referring to Figures 4 and the proof of the last theorem we see that

$$\phi = \tfrac{1}{2}\pi + a - \theta_0,$$

$$\cot\phi = \tan(\theta_0 - a) = \frac{f'(\theta_0)}{f(\theta_0)} = \frac{dr}{rd\theta}.$$

This formula is perfectly general, so that we have in all cases the important formula

(2·732) $$\tan\phi = \frac{rd\theta}{dr}.$$

To obtain a formula for the differential of the arc, suitable for polar coordinates, we have in all cases

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and therefore

$$dx = dr\cos\theta - r\,d\theta\sin\theta,$$

$$dy = dr\sin\theta + r\,d\theta\cos\theta;$$

$$(ds)^2 = (dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2,$$

(2·733) $$ds = \pm\{(dr)^2 + r^2(d\theta)^2\}^{\frac{1}{2}} = +\{r^2 + (dr/d\theta)^2\}^{\frac{1}{2}}\,d\theta,$$

$$\frac{ds}{d\theta} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

We have also in all cases

(2·734) $$\frac{rd\theta}{ds} = \sin\phi^*, \quad \frac{dr}{ds} = \cos\phi.$$

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* Note that ϕ can only be greater than π if r is negative.

§ 2·80. **Concavity and convexity.** Consider a curve given by the equation $y=f(x)$ in rectangular *or oblique* coordinates, and a point P on the curve where $f'(x)$ exists, *i.e.* where the curve has a tangent not parallel to the axis of y. There are two sides to this tangent, which are distinguished by the facts that on one side $y\to+\infty$ along any line parallel to the y-axis, while on the other $y\to-\infty$. We shall call the side on which $y\to+\infty$ the **upper** side, and the side on which $y\to-\infty$ the **lower** side, of this line (the tangent). *If the curve lies entirely on one side of the tangent in the neighbourhood of P, it is said to be* **concave upwards** *or* **convex upwards at** P* *according as it lies on the upper or the lower side of the tangent in the neighbourhood of P.*

THEOREM 2·81. *The curve $y=f(x)$, for which $f(x)$ possesses a continuous second differential coefficient at P, is concave (convex) upwards at P if*

$$f''(x)>0\ (<0).$$

The curve will be concave or convex upwards (see § 2·40) according as $\delta y-dy$ is positive or negative for all sufficiently small values of δx of either sign. But, by Taylor's theorem,

$$(2·811)\qquad \delta y-dy=\tfrac{1}{2}(\delta x)^2 f''(x+\theta\,\delta x),\ (0<\theta<1).$$

As $f''(x)$ is continuous, the theorem follows at once.

A curve $y=f(x)$ is said to be concave (convex) upwards throughout the interval $x_0\leqslant x\leqslant x_1$, if every point (excluding the end points) of any arc of the curve in this interval lies below (above) the corresponding chord.

This is equivalent to the statement that the curve is concave (convex) upwards throughout the interval, if

$$(2·82)\qquad f(\xi_1)+f(\xi_3)-2f(\xi_2)>0\ (<0)$$

for all values of ξ_1, ξ_2, ξ_3, such that

$$x_0\leqslant\xi_1<\xi_2<\xi_3\leqslant x_1.$$

It is easily verified that, if $f''(x)$ is continuous throughout the interval, 2·82 is equivalent to the condition

$$(2·821)\qquad f''(x)>0\ (<0),\ (x_0\leqslant x\leqslant x_1).$$

It follows at once by comparison of 2·811 and 2·821 that *if such a curve is concave (convex) upwards throughout an interval it is concave (convex) upwards at every point of the interval and conversely.*

---

* Alternatively, convex downwards and concave downwards respectively. It should be remembered that "upwards" is used simply to denote the direction in which $y\to+\infty$. This is because the positive axis of y is usually so drawn, but the word "upwards" must be applied in accordance with the above definition whatever the direction actually chosen for the positive axis of y.

Curves that are concave (convex) upwards throughout an interval are of considerable importance in analysis; their properties can however be discussed without any hypothesis as to the existence of $f''(x)$. Their further treatment is out of place here*.

§ 2·90. **Points of inflexion.** DEFINITION. *A* **point of inflexion** *is a point at which the curve crosses its tangent.*

For the curve $y=f(x)$, with a continuous $f''(x)$, we have

$$\delta y - dy = \tfrac{1}{2} f''(x+\theta\delta x)(\delta x)^2, \quad (0<\theta<1).$$

In order that x may be a point of inflexion for such a curve, it is necessary and sufficient that $\delta y - dy$ should change sign with δx, *i.e.* that

$$f''(x+\mu)$$

should change sign with μ. *Hence a point of inflexion is a root of $f''(x)=0$ at which $f''(x)$ changes sign, and conversely.* A sufficient condition that $f''(x)$ should change sign is that, if $f^{(n)}(x)$ is the first differential coefficient not vanishing with $f''(x)$, n should be odd. For we have

$$\delta y - dy = \frac{(\delta x)^n}{n!} f^{(n)}(x+\theta\delta x), \quad (0<\theta<1).$$

Developments of this nature belong more properly to the theory of contact.

It must be remembered, however, that a curve may have a point of inflexion at a point at which the tangent is parallel to the y-axis, and at such a point $f''(x)$ does not exist. To avoid this case of exception, we may say that, for a curve for which one at least of d^2y/dx^2 and d^2x/dy^2 exists and is continuous, the points of inflection are the points at which one at least of d^2y/dx^2 and d^2x/dy^2 vanishes and changes sign.

Consider now the curve

$$x=\phi_1(t), \quad y=\phi_2(t),$$

for which $\phi_1''(t)$, $\phi_2''(t)$ are continuous. The equation of the tangent at t_0 is

$$(x-\phi_1(t_0))\,\phi_2'(t_0)-(y-\phi_2(t_0))\,\phi_1'(t_0)=0.$$

The perpendicular distance of (x, y), a point on the curve, from the tangent is algebraically proportional to

$$\begin{aligned}(\phi_1(t)-\phi_1(t_0))\,\phi_2'(t_0)-(\phi_2(t)-\phi_2(t_0))\,\phi_1'(t_0)\\ =\tfrac{1}{2}(t-t_0)^2\begin{vmatrix}\phi_1''(t_0+\mu) & \phi_2''(t_0+\mu)\\ \phi_1'(t_0) & \phi_2'(t_0)\end{vmatrix},\end{aligned}$$

---

* See d.l.V.P., Vol. I, pp. 285–291.

where $\mu = \theta (t - t_0)$ and $0 < \theta < 1$. The curve therefore crosses the tangent if

$$\text{(2·901)} \qquad \begin{vmatrix} \phi_1''(t) & \phi_2''(t) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{vmatrix}$$

vanishes and changes sign at $t = t_0$.

§ 2·910. **Concavity, convexity and points of inflexion in polar coordinates.** The property of being concave or convex *to a point* may be defined in a similar manner to the property of the being concave or convex upwards. If the curve lies entirely on one side of the tangent at P, and Q is a point not lying on the tangent at P, we say that *the curve is concave (convex) at P to the point Q according as the curve near P lies on the same (opposite) side of the tangent at P as the point Q.*

Take the point Q for the origin of polar coordinates, and suppose that the curve is $r = f(\theta)$, and that $f''(\theta)$ is continuous. The equation of the tangent is

$$\frac{f(\theta_0)}{r} = \cos(\theta - \theta_0) - \frac{f'(\theta_0)}{f(\theta_0)} \sin(\theta - \theta_0).$$

The curve will be concave (convex) to the origin if

$$\frac{f(\theta_0)}{f(\theta)} - \cos(\theta - \theta_0) + \frac{f'(\theta_0)}{f(\theta)} \sin(\theta - \theta_0) > 0 \; (< 0),$$

when $|\theta - \theta_0|$ is small. This condition reduces to

$$(f(\theta_0))^2 + 2(f'(\theta_0))^2 - f(\theta_0) f''(\theta_0) > 0 \; (< 0).$$

In the same way the condition for a point of inflexion for such a curve is that

$$(f(\theta))^2 + 2(f'(\theta))^2 - f(\theta) f''(\theta)$$

should vanish and change sign at $\theta = \theta_0$.

Much of what proceeds takes a simpler form if the curve is expressed as

$$u = f(\theta),$$

where $u = 1/r$. The last expression in the condition for an inflexion is then replaced by

$$f(\theta) + f''(\theta).$$

EXAMPLES I

(1) Prove that the μ of § 2·620 satisfies the relation

$$\mu \sim \tfrac{1}{8} f''(x_0)(\delta x)^2$$

for the curve $y = f(x)$, and deduce that $\mu = o(\delta x)^2$ implies that the curve is a straight line.

[At the point of greatest distance between arc and chord, the tangent must be parallel to the chord. Hence take the tangent and normal as axes and consider the intersections of $y=f(x)$ $(f(0)=f'(0)=0)$ and $y=\mu$.]

(2) Taking $x=f(t)$, $y=g(t)$ for the curve and writing

$$\Omega^2=(f'(t))^2+(g'(t))^2,$$

prove that

$$\text{arc}-\text{chord}\sim\frac{(\delta t)^3}{24\,\Omega^3}(g''f'-f''g')^2$$

$$=(\delta s)^3/24\rho^2. \qquad \text{(see 3·132)}$$

Deduce that

$$\text{arc}-\text{chord}=o\,(\delta s)^3$$

implies that the curve is a straight line.

[We have

$$\delta\,(\text{arc})=\int_0^{\delta t}\{(f'(t))^2+(g(t))^2\}^{\frac{1}{2}}dt,$$

$$\delta\,(\text{chord})=\{(f(t+\delta t)-f(t))^2+(g\,(t+\delta t)-g\,(t))^2\}^{\frac{1}{2}}.$$

Expand by Taylor's theorem in powers of δt as far as $(\delta t)^3$, and the result follows. If $(\text{arc}-\text{chord})=o\,(\delta t)^3$, then $f''/f'=g''/g'$. Integrating, $f'=Ag'$, $f=Ag+B$, which defines a straight line.]

(3) Deduce $\mu=O\,(\delta s)^2$ directly from

$$\text{arc}-\text{chord}=O\,(\delta s)^3.$$

[Establish and use the fact that

$$\mu^2<\tfrac{1}{4}\{(\text{arc})^2-(\text{chord})^2\}.]$$

(4) Using polar coordinates, prove that, if $f'(\theta)$ is continuous at $\theta=\theta_0$, the tangent is the limit of the chord Q_1Q_2 as Q_1, $Q_2\rightarrow P\,(r_0,\,\theta_0)$.

[Combine the proofs of 2·71 and 2·21.]

(5) With the notation of Theorem 2·624 prove that, in order that $PT/QT\rightarrow 1$, it is sufficient that $f''(x)$ should be continuous and not zero at P.

[Take the tangent and normal as axes, and the curve as

$$y=f(x),\quad (f(0)=f'(0)=0).$$

The tangent at x is given by 2·11. Find the coordinates of T. Theorem 2·624 itself may be established similarly. Note that the condition $f''(x)\neq 0$ is invariant, § 1·40.]

(6) Three neighbouring tangents are drawn to a curve. If δs_1, δs_2 be the lengths of the arcs between the points of contact taken in order along the curve, $\delta\psi$ the angle between the extreme tangents, and Δ the area of the enclosed triangle, then

$$\Delta\sim\tfrac{1}{8}\,\delta s_1\,\delta s_2\,\delta\psi.$$

It is assumed that the points of contact tend to a point at which $f''(x)$ is continuous and not zero.

[The lengths of the two sides including the angle $\delta\psi$ are asymptotically equivalent to $\frac{1}{2}\delta s_1$ and $\frac{1}{2}\delta s_2$ respectively. This follows at once from Exs. 3 and 5.]

(7) Prove geometrically that, if PT the tangent at P exists, then the chord Q_1Q_2 tends to the tangent at P as Q_1, $Q_2 \to P$ from opposite sides. Show where the proof fails when Q_1, $Q_2 \to P$ in any manner.

Show that if PQ_1, PQ_2 are equal and opposite arcs, and $f'''(x)$ exists at P, the chord Q_1Q_2 represents the tangent at P with an error $O(\delta s)^2$: and that some condition of equality is essential.

[The angle between PT and Q_1Q_2 is less than the angle between PT and PQ_1. Hence Q_1Q_2 tends to parallelism with PT, etc.]

(8) Define a tangent of the twisted curve

$$x=\phi_1(t),\ y=\phi_2(t),\ z=\phi_3(t),$$

and obtain its equations in the forms

$$\frac{x-x_0}{\phi_1'}=\frac{y-y_0}{\phi_2'}=\frac{z-z_0}{\phi_3'},$$

$$\frac{x-x_0}{dx}=\frac{y-y_0}{dy}=\frac{z-z_0}{dz}.$$

If l_1, m_1, n_1 are the direction cosines of the tangent, prove that

$$dx=l_1ds,\ dy=m_1ds,\ dz=n_1ds.$$

[d.l.V.P., Vol. I, pp. 325–331.]

(9) The osculating plane of a twisted curve being defined as the limit of the plane PQ_1Q_2 when Q_1 and Q_2 tend to P along the curve, obtain its equation in the forms

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix}=0, \qquad \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix}=0.$$

Prove that the osculating plane is the limit of

(α) a plane through the tangent at P and the chord PQ,

(β) a plane through the tangent at P and parallel to the tangent at Q. Alternatively, show that the angle between the normal to the osculating plane at P and the tangent at Q is $\frac{1}{2}\pi+o(\delta s)$, if ϕ_1'' etc. exist at P, and is $\frac{1}{2}\pi+O(\delta s)^2$ if ϕ_1''' etc. exist at P.

[d.l.V.P., Vol. I, pp. 335–337.]

(10) If PQ_1, PQ_2 are equal and opposite arcs δs, prove that under suitable conditions the plane PQ_1Q_2 represents the osculating plane at P with an error $O(\delta s)^2$, but that if the arcs are not equal the error will usually be $O(\delta s)$.

(11) The line of intersection of the osculating plane at P with the osculating plane at Q tends to the tangent at P as $Q \to P$.

[d.l.V.P., Vol. I, p. 346: or geometrically, using Ex. 10.]

CHAPTER III

THE CURVATURE OF PLANE CURVES

§ 3·10. **Curvature.** The idea of curvature is introduced to afford a measure of the rate at which the tangent is turning as the point of contact moves along the curve. Suppose that PQ are two points on any rectifiable curve which has a tangent at every point of the arc PQ, δs the length of the arc PQ, and $\delta\psi$ the angle between the tangents at P and Q. Then $\delta\psi/\delta s$ is called *the mean curvature of the arc* PQ, and

$$\operatorname*{Lt}_{Q\rightarrow P} \frac{\delta\psi}{\delta s},$$

if it exists, is called *the curvature at* P. If this limit is denoted by $1/\rho$, ρ is called *the radius of curvature at* P*.

We shall find in practice that it is necessary to attach a *sign* to the mean curvature, curvature, and radius of curvature. Consider the case of two equal circles touching externally at P. The mean curvatures of the equal arcs PQ_1, PQ_2 are, as at present defined, equal, though the tangents turn in opposite directions as the points of contact go from P to Q_1 and Q_2. To take account of this we introduce the following definitions.

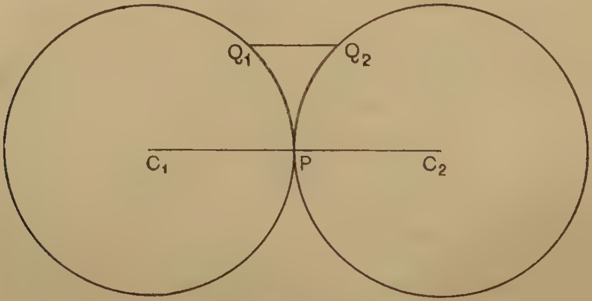

Fig. 5.

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* If the curvature at P is zero, there is no radius of curvature or "an infinite radius of curvature" according to choice.

DEFINITIONS. *With the conventions of* § 2·60, **the mean curvature of the arc** *PQ is defined as*

$$(\psi_Q - \psi_P)/(s_Q - s_P),$$

the arc s being measured from any convenient point on the curve, and the **curvature** *at P is defined as*

$$\operatorname*{Lt}_{Q \to P} (\psi_Q - \psi_P)/(s_Q - s_P),$$

if this limit exists.

The radius of curvature at *P is the reciprocal of the curvature, and is usually denoted by ρ.*

The centre of curvature at *P is a point C on the normal at P such that $PC = \rho$**.

The circle of curvature at *P is a circle with centre C and radius $PC\,(=\rho)$.*

We find in all cases that the coordinates (X, Y) of C satisfy

$$X = x - \rho \sin\psi, \quad Y = y + \rho\cos\psi; \tag{3·11}$$

or, if (l', m') are the direction cosines of the positive normal,

$$X = x + l'\rho, \quad Y = y + m'\rho. \tag{3·12}$$

THEOREM 3·13. *The necessary and sufficient condition that the curve $y = f(x)$ should have a definite curvature at any point x, where the tangent is not parallel to Oy, is that $f''(x)$ should exist. The curvature $1/\rho$ is then given by*

$$\frac{1}{\rho} = \frac{f''(x)}{\{1 + (f'(x))^2\}^{\frac{3}{2}}}. \tag{3·131}$$

(1) The condition is necessary. For in order that ψ_P and ψ_Q may exist, it is necessary that the curve should have a tangent in the neighbourhood of P; and in order that $\operatorname{Lt}(\psi_Q - \psi_P)/(s_Q - s_P)$ may exist it is necessary that $\psi_Q \to \psi_P$, *i.e.* that ψ should be continuous at P. As

$$\tan\psi = f'(x), \quad (\psi \neq \tfrac{1}{2}\pi), \tag{3·1311}$$

$f'(x)$ must be continuous at P.

If Q is the point $(x + \delta x, y + \delta y)$,

$$\delta s \doteq \int_x^{x+\delta x} \{1 + (f'(\xi))^2\}^{\frac{1}{2}}\, d\xi.$$

Since $f'(\xi)$ is continuous at $\xi = x$,

$$\{1 + (f'(\xi))^2\}^{\frac{1}{2}} = \{1 + (f'(x))^2\}^{\frac{1}{2}} + o(1),$$

$$\delta s = \delta x\,\{1 + (f'(x))^2\}^{\frac{1}{2}} + o(\delta x).$$

---

* Of course with due regard to the sign of ρ and the positive direction of the normal Pn. If ρ is positive C lies on the positive normal Pn; if ρ is negative C lies on Pn'.

Therefore $$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta x}{\delta s} = \{1 + (f'(x))^2\}^{-\frac{1}{2}} \neq 0.$$

Also $$\delta\psi/\delta s = (\delta\psi/\delta x) \times (\delta x/\delta s),$$

and therefore, as $\delta\psi/\delta s$ has a limit, and $\delta x/\delta s$ a non-zero limit, as $\delta x \to 0$, $\delta\psi/\delta x$ must have a limit. But applying the mean value theorem to 3·1311 we have

$$\delta\psi \sec^2(\psi + \theta'\delta\psi) = f'(x + \delta x) - f'(x), \quad (0 < \theta' < 1),$$

$$\frac{\delta\psi}{\delta x} = \frac{f'(x+\delta x) - f'(x)}{\delta x} \cos^2(\psi + \theta'\delta\psi).$$

But $\cos^2(\psi + \theta'\delta\psi)$ has the non-zero limit $(1 + \{f'(x)\}^2)^{-1}$, so that

$$\{f'(x+\delta x) - f'(x)\}/\delta x$$

must have a limit, *i.e.* $f''(x)$ must exist.

(2) Conversely, the above reasoning may be reversed, and if $f''(x)$ exists, then

$$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta\psi}{\delta x} = \frac{f''(x)}{1 + \{f'(x)\}^2},$$

$$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta x}{\delta s} = (1 + \{f'(x)\}^2)^{-\frac{1}{2}};$$

so that $$\operatorname*{Lt}_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \operatorname*{Lt}_{\delta x \to 0} \frac{\delta\psi}{\delta x} \times \operatorname*{Lt}_{\delta x \to 0} \frac{\delta x}{\delta s};$$

and finally $$\frac{1}{\rho} = \frac{f''(x)}{(1 + \{f'(x)\}^2)^{\frac{3}{2}}}\text{*}.$$

At points at which $d\psi/ds = 0$, the curve is said to have *a stationary tangent*. It is easy to see that, if x_0 is such a point, $f''(x_0) = 0$ provided $f''(x_0)$ exists. At all points of inflexion, therefore, at which $f''(x)$ exists, the curve has a stationary tangent, but the converse is not true, for at a point of stationary tangent the curve may not cross its tangent. The further consideration of such points belongs more properly to the theory of singular points.

Other formulae for the curvature $1/\rho$ are

(3·132) $$\frac{1}{\rho} = \frac{\phi_1'\phi_2'' - \phi_2'\phi_1''}{(\phi_1'^2 + \phi_2'^2)^{\frac{3}{2}}},$$

when the curve is $x = \phi_1(t)$, $y = \phi_2(t)$; and

(3·133) $$\frac{1}{\rho} = -\frac{F_{xx}'' F_y'^2 - 2F_{xy}'' F_x' F_y' + F_{yy}'' F_x'^2}{(F_x'^2 + F_y'^2)^{\frac{3}{2}}},$$

---

* If $Q \to P$ from the right (left) then $f''(x)$ need not exist, but it is necessary and sufficient that $f'(x)$ should have a differential coefficient on the right (left).

when the curve is $F(x, y)=0$. These are direct deductions from (3·131) and the theorems of the differential calculus. The proofs may be left to the reader.

§ 3·20. **Properties of the centre of curvature.** The centre of curvature C has a large number of properties many of which give rise to alternative definitions. These are contained in the following theorems.

THEOREM 3·21. *A curve is always concave to its centre of curvature.*
The proof of this theorem is left to the reader.

THEOREM 3·22. *If the normal at Q cuts the normal at P in Γ, and C the centre of curvature at P exists, then $\Gamma \to C$ as $Q \to P$.*

Take as axes of coordinates the tangent and normal to the curve at P. By § 1·40 this does not affect the generality of the argument. Then, by Theorem 1·52, the equation of a sufficiently small arc of the curve containing P can be put in the form

$$y=f(x), \quad (|x|<\delta),$$

where $f'(0)=0$ and $f''(0)$ exists and is not zero*. The coordinates of C are $(0, \rho)$ or $(0, 1/f''(0))$. The equation of the normal at $Q(\xi, f(\xi))$ is

$$f'(\xi)(y-f(\xi))+(x-\xi)=0,$$

which cuts $x=0$ where

$$y=f(\xi)+\xi/f'(\xi).$$

This is the ordinate of Γ. As $Q \to P$, $\xi \to 0, f(\xi) \to 0$, and

$$\xi/f'(\xi)=(\xi-0)/(f'(\xi)-f'(0)) \to 1/f''(0)$$

by definition, since $f''(0)$ exists *and is not zero*. Hence $\Gamma \to C$.

THEOREM 3·23. (A) *If the normal at Q_1 cuts the normal at Q_2 in Γ, and C the centre of curvature at P exists, then $\Gamma \to C$ as $Q_1 \to P$ and $Q_2 \to P$, if Q_1 and Q_2 are always on opposite sides of P.*

(B) *But if $Q_1 \to P$ and $Q_2 \to P$ in any manner, Γ need not tend to C unless ρ is continuous at P.*

(A) Let $Q_1\Gamma$ cut PC in Γ_1, $Q_2\Gamma$ cut PC in Γ_2; then, if Q_1 and Q_2 are always on opposite sides of P, Γ cannot lie on $Q_1\Gamma_1$ and $Q_2\Gamma_2$ or on both $Q_1\Gamma_1$ and $Q_2\Gamma_2$ produced; it can only lie on one of them and on the other produced, as in the figure. It is true that Γ, Γ_1, Γ_2 can all coincide, but this will not affect the succeeding argument. As $Q_1\Gamma Q_2 \to 0$, $\Gamma_1\Gamma\Gamma_2$ is ultimately greater than $\frac{1}{2}\pi$, and therefore Γ lies inside a circle on $\Gamma_1\Gamma_2$ as diameter. But $\Gamma_1 \rightarrow C$, $\Gamma_2 \rightarrow C$ as $Q_1 \rightarrow P$, $Q_2 \rightarrow P$; it therefore follows that $\Gamma \rightarrow C$ as Q_1 and Q_2 tend to P from opposite sides.

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* The existence of C implies that C is a finite point according to our usage.

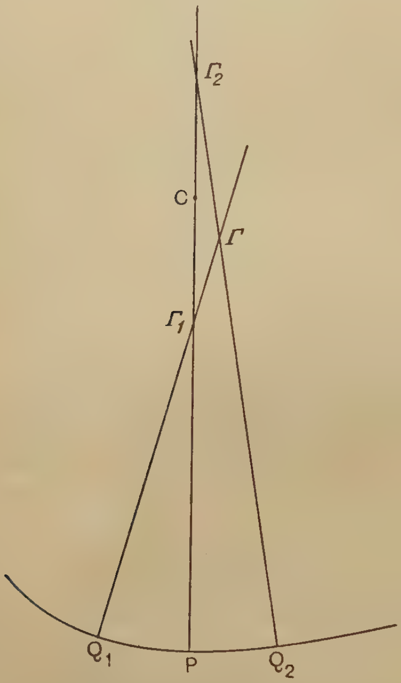

Fig. 6 (*a*).

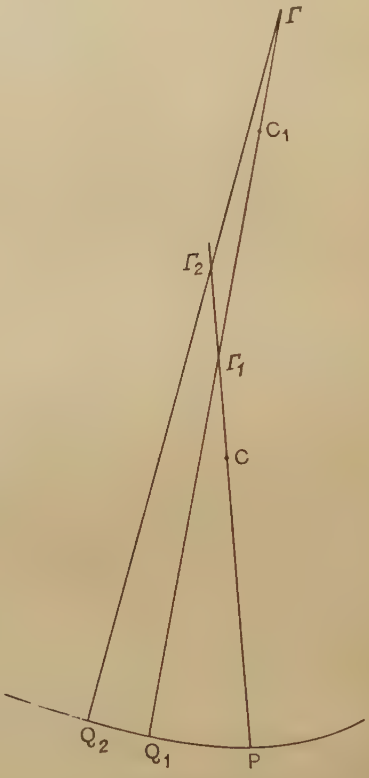

Fig. 6 (*b*).

(B) The preceding argument breaks down here. But if ρ is continuous at P, there must exist a centre of curvature C_1 at Q_1 if Q_1 is sufficiently near P. By the preceding reasoning, if Q_1 is the middle point of the three Q_2, Q_1, P, then Γ, Γ_1, $\Gamma_2 \rightarrow C_1$ as $Q_2 \rightarrow Q_1$, $P \rightarrow Q_1$, Q_1 remaining fixed. In other words, given ϵ we can find a number δ_1 such that $\Gamma C_1 < \epsilon$ provided only $Q_2Q_1 < \delta_1$, $PQ_1 < \delta_1$. But since ρ is continuous at P, $C_1 \rightarrow C$ as $Q_1 \rightarrow P$, or in other words there exists a number δ_2 such that $C_1C < \epsilon$ if only $Q_1P < \delta_2$. Hence there exists a number δ (the lesser of δ_1 and δ_2) such that $C\Gamma < 2\epsilon$ if only $Q_1P < \delta$, $Q_2P < \delta$. In other words $\Gamma \rightarrow C$ as required.

On the other hand it is possible to construct cases in which, if ρ is not continuous at P, Γ does not tend to C as $Q_1, Q_2 \to P$ in a suitable manner.

This theorem is an example of the general principle explained in Note A to § 1·30 of the Introduction.

THEOREM 3·24. *If a circle whose centre is O cut* a regular arc of any curve in two points Q_1, Q_2, then there exists a point P on the curve between† Q_1 and Q_2 such that OP is a normal to the curve.*

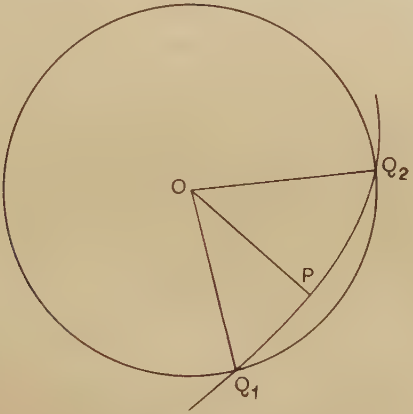

Fig. 7.

By a "regular" arc we mean an arc along which the curve can be put in the general form

$$x = \phi_1(t), \quad y = \phi_2(t),$$

where $\phi_1'(t)$, $\phi_2'(t)$ exist, and are not both zero, at Q_1, Q_2 and all points of the arc between.

We have, if O is the point (a, b), and P any point on the arc,

$$OP^2 = (a - \phi_1(t))^2 + (b - \phi_2(t))^2;$$

therefore OP^2 is a continuous function of t while P lies on the arc Q_1Q_2, possessing everywhere a differential coefficient

$$-2\{(a - \phi_1)\phi_1' + (b - \phi_2)\phi_2'\},$$

and taking the same value (R^2 say) both at Q_1 and Q_2. Therefore by

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* Nothing prevents the curve and circle from *touching* at one or both of Q_1 and Q_2.

† *I.e.* a value of t such that $t_1 < t < t_2$ if t_1 and t_2 are the parameters of the points Q_1, Q_2.

Rolle's Theorem, there exists a point P between Q_1 and Q_2* such that $\frac{d}{dt}(OP^2)=0$, *i.e.* such that

$$(a-\phi_1)\,\phi_1'+(b-\phi_2)\,\phi_2'=0.$$

This is satisfied at any point at which ϕ_1' and ϕ_2' both vanish. But if as we have presupposed they do not, it is the condition that OP is normal to the curve at the point P as required.

We can now prove at once two more theorems embodying possible definitions of C.

THEOREM 3·25. *If a circle centre O be drawn touching the curve at P and cutting it at Q, and if C the centre of curvature at P exists, then $O \to C$ as $Q \to P$.*

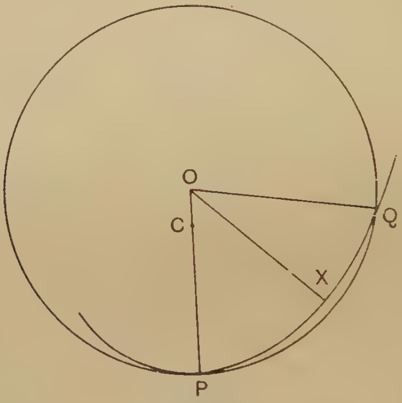

Fig. 8.

Since C exists, the conditions of Theorem 3·24 are satisfied when Q is sufficiently close to P. Therefore there exists a point X between Q and P such that OX is normal to the curve. As $Q \to P$, $X \to P$, and as $X \to P$, $O \to C$, by Theorem 3·22, as was to be proved.

In a similar manner we prove

THEOREM 3·26. *If a circle centre O be drawn cutting the curve at three points Q_1, Q_2, Q_3, and if ρ is continuous at P, then $O \to C$ as $Q_1, Q_2, Q_3 \to P$*†.

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* *I.e.* a value of t such that $t_1 < t < t_2$ if t_1 and t_2 are the parameters of the points Q_1, Q_2.

† Note the following enunciation which does not involve the hypothesis of continuity:—

If a circle centre O be drawn cutting the curve at three points Q_1, P, Q_2 and C, the centre of curvature at P, exists, then $O \to C$ as Q_1, $Q_2 \to P$ from opposite sides.

THEOREM 3·27. *The osculating circle, when it exists, is the same as the circle of curvature.*

This is a particular case of theorems on contact. See § 4·40.

Each of the foregoing theorems embodies a possible definition of C the centre of curvature at a point P, but it is clear that some are more general than others. Thus any of the definitions embodied in Theorems 3·22, 3·23 (A), 3·25, are as general as our definition, and accordingly may replace it. The others are not so general and so are not suitable. Conversely, it may be shown that if any of these alternative definitions be adopted, then, at any point at which the curve has a curvature $1/\rho$ by the new definition,

$$\operatorname*{Lt}_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{1}{\rho}.$$

It follows that these definitions are completely equivalent. As a specimen of the theorems to be proved we take the following:

If the normal at Q cuts the normal at P in Γ, and if Γ tends to a finite limit C as $Q \to P$, then

$$\operatorname*{Lt}_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{1}{PC},$$

with the proper conventions of sign.

This is the converse of Theorem 3·22, and the proof is simply the proof of that theorem turned backwards, with obvious changes.

There are a variety of other circles which tend to the circle of curvature at P as a limit. The following are instances.

*A circle touching the curve at P, and the tangent at a neighbouring point Q**.

*A circle touching the tangent at P** *and the tangents at neighbouring points Q_1, Q_2*†.

A circle passing through P and touching two tangents at neighbouring points Q_1, Q_2‡.

A circle touching the tangent at P, and passing through two neighbouring points Q_1, Q_2‡.

In each case P may be replaced by a point Q_3 which tends to P. Direct proofs of all these theorems are not difficult (with the obvious assumptions). They depend on a combination of Theorems 3·25 and 3·26 with well-known properties of triangles.

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* *I.e.*, the circle touches Qt, the tangent to the curve at Q, but touches Qt not necessarily at Q.

† There are four circles touching these three tangents, of which three have as limit the point P.

‡ There are two such circles in each case, both of which have the circle of curvature at P as limit.

§ 3·30. **The closeness of approximations to the circle of curvature.** It is often useful to know the order of the error involved in replacing the circle of curvature by one of the circles which has the circle of curvature for a limit, for instance the circle through P and two neighbouring points Q_1, Q_2.

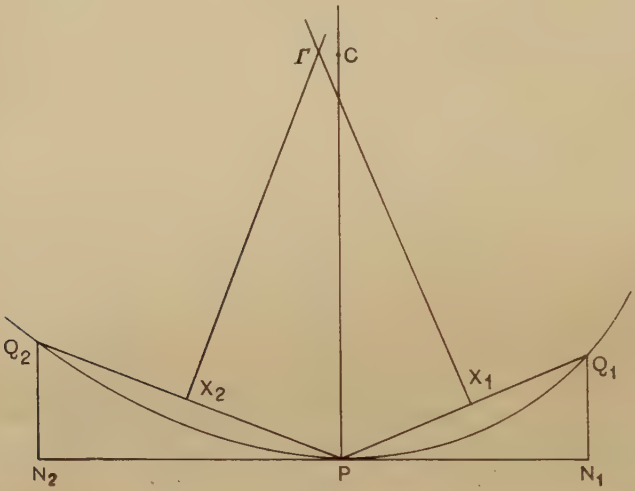

Fig. 9.

Taking the tangent and normal at P for axes, the curve is

$$y=f(x), \quad (f(0)=f'(0)=0, \quad f''(0)\neq 0);$$

C is the point $(0, 1/f''(0))$, and Γ the intersection of the perpendicular bisectors of PQ_1, PQ_2. Then Γ lies on the two lines

$$(3{\cdot}31) \qquad \begin{cases} \xi_1 x + \eta_1 y - \frac{1}{2}(\xi_1^2+\eta_1^2)=0, \\ \xi_2 x + \eta_2 y - \frac{1}{2}(\xi_2^2+\eta_2^2)=0, \end{cases}$$

where Q_1 is (ξ_1, η_1) and Q_2 is (ξ_2, η_2). If $f^{\text{iv}}(0)$ exists, we have

$$\eta_1=\tfrac{1}{2}\xi_1^2 f''(0)+\tfrac{1}{6}\xi_1^3 f'''(0)+O(\xi_1^4),$$

and therefore the equations (3·31) reduce to

$$x+\{\tfrac{1}{2}\xi_1 f''(0)+\tfrac{1}{6}\xi_1^2 f'''(0)+O(\xi_1^3)\}\,y-\tfrac{1}{2}\xi_1(1+O(\xi_1^2))=0,$$

$$x+\{\tfrac{1}{2}\xi_2 f''(0)+\tfrac{1}{6}\xi_2^2 f'''(0)+O(\xi_2^3)\}\,y-\tfrac{1}{2}\xi_2(1+O(\xi_2^2))=0.$$

Solving, we have

$$\frac{x}{O(\xi^3)} = \frac{y}{\frac{1}{2}(\xi_2-\xi_1)+O(\xi^3)} = \frac{1}{\frac{1}{2}(\xi_2-\xi_1)f''(0)+\frac{1}{6}(\xi_2^2-\xi_1^2)f'''(0)+O(\xi^3)},$$

where ξ denotes the numerically greater of ξ_1 and ξ_2.

It follows that in general the ordinates of C and Γ differ by a length of order ξ, but that if $\xi_2 = -\xi_1$ then

$$(3\cdot32) \qquad x = O(\xi^2),\ y = \frac{1}{f''(0)} + O(\xi^2).$$

We have therefore proved that *if the arcs* PQ_1, PQ_2 *are equal and opposite* (*say* $\pm\delta s$), *or more generally if* $PQ_1 = \delta s$, $PQ_2 = -\delta s + O(\delta s)^2$, *then*

$$(3\cdot33) \qquad \Gamma C = O(\delta s)^2.$$

Since both circles pass through P, their radii also differ by $O(\delta s)^2$, and in fact *the circle* Q_2PQ_1 *represents the circle of curvature at* P *with an error* $O(\delta s)^2$.

This fact is important if we wish to apply geometrical reasoning to approximate figures. If, for instance, we are going to argue about C and C_1 (the centre of curvature at Q_1), representing them in the foregoing manner by Γ and Γ_1, $\Gamma\Gamma_1$ or CC_1 will itself be a small quantity of the first order and, unless ΓC and $\Gamma_1 C_1$ are of higher order than the first, no such argument can possibly be legitimate. It is essential to take the precaution indicated by the preceding discussion*. When this has been done we have for instance

$$\Gamma\Gamma_1/CC_1 = 1 + O(\delta s),$$

whereas in the other case it is by no means evident even that

$$\text{Lt}\ \Gamma\Gamma_1/CC_1 = 1.$$

In a similar way it may be shown that, if K is the intersection of normals at Q_1 and Q_2,

$$(3\cdot34) \qquad KC = O(\delta s)^2$$

if and only if PQ_1 and PQ_2 are, with a possible error $O(\delta s)^2$, equal and opposite arcs.

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* Similar arguments must be applied in the theory of twisted curves when approximate geometrical figures are used. The use of such figures can only be legitimate when the errors in the approximate representations are of the second or higher order. This point appears to be often overlooked.

§ 3·40. **Newton's Method.** A formula that is often useful is contained in the following theorem.

THEOREM 3·41. *If the tangent and normal at P are taken as axes of x and y, and ρ exists, then*

$$\rho = \operatorname*{Lt}_{x \to 0} \frac{x^2}{2y}. \tag{3·42}$$

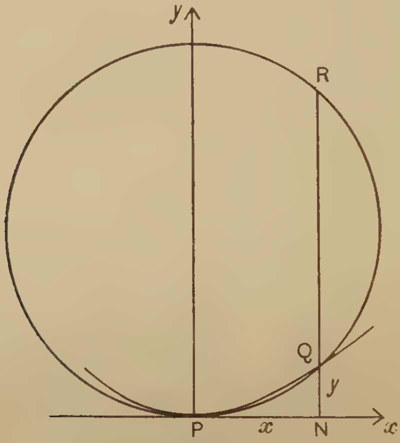

Fig. 10.

Let Q be the point (x, y). Draw a circle to touch the curve at P and pass through Q; then, by Theorem 3·25, this circle tends to the circle of curvature as $Q \to P$, *i.e.* as $x \to 0$. If R is the radius of this circle, $RN = 2R - y$. We have also

$$NP^2 = NQ \, . \, NR,$$

so that
$$x^2/y = 2R - y \to 2\rho,$$

which proves the theorem.

§ 3·50. **The Differentials** *dl*, *dm*, **etc.** Theorem 3·22 has an important consequence. The equation of the normal at (ξ, η) may be written

$$(x - \xi)\, l + (y - \eta)\, m = 0, \tag{3·501}$$

where (l, m) are as usual the direction cosines of the tangent at (ξ, η). $C\,(X, Y)$ lies on this line, so that

$$(X - \xi)\, l + (Y - \eta)\, m = 0. \tag{3·502}$$

But C is the limit of intersections of neighbouring normals, and therefore* (X, Y) also satisfy the equation

$$(3\cdot 503) \qquad (X-\xi)\,dl + (Y-\eta)\,dm = l\,d\xi + m\,d\eta = ds.$$

But we have also $l^2+m^2=1$, and therefore

$$l\,dl + m\,dm = 0.$$

Hence dl and dm satisfy

$$(3\cdot 504) \qquad \begin{cases} (Y-\eta)\,dl - (X-\xi)\,dm = 0, \\ (X-\xi)\,dl + (Y-\eta)\,dm = ds; \end{cases}$$

which may also be written

$$(3\cdot 505) \qquad \begin{cases} m'dl - l'dm = 0, \\ l'dl + m'dm = ds/\rho, \end{cases}$$

where (l', m') are the direction cosines of the positive normal.

We have therefore

$$(3\cdot 51) \qquad \frac{dl}{ds} = \frac{l'}{\rho}, \quad \frac{dm}{ds} = \frac{m'}{\rho},$$

$$(3\cdot 52) \qquad \frac{1}{\rho^2} = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2.$$

Remembering that $l = dx/ds$, $m = dy/ds$, we have

$$(3\cdot 53) \qquad \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

With our conventions†, l' and m' are connected with l and m by the relations

$$l' = m, \quad m' = -l.$$

---

* The full reasoning here is as follows. (X, Y) is the limit of the intersection of

$$(x-\xi)\,l + (y-\eta)\,m = 0$$

and
$$(x-\xi-\delta\xi)(l+\delta l) + (y-\eta-\delta\eta)(m+\delta m) = 0,$$

i.e. of
$$(x-\xi)\,l + (y-\eta)\,m = 0$$

and
$$(x-\xi)\,\delta l + (y-\eta)\,\delta m = \{(l+\delta l)\,\delta\xi + (m+\delta m)\,\delta\eta\}.$$

Now when $f''(x)$ exists $d\psi$ exists, and therefore dl and dm exist. Therefore (X, Y) satisfy

$$(x-\xi)\,dl + (y-\eta)\,dm = ds,$$

which is the limiting form of the second equation.

† In general (l', m') satisfy

$$l'^2 + m'^2 = 1, \quad ll' + mm' = 0$$

(condition of perpendicularity), and therefore

$$l' = \lambda m, \quad m' = -\lambda l,$$

where $\lambda = \pm 1$. $\lambda = 1$ when the conventions are chosen so that, the positive tangent coinciding with Ox, the positive normal coincides with Oy; $\lambda = -1$ when, in like case, the positive normal coincides with Oy'.

Therefore

$$\frac{dl'}{ds}=-\frac{l}{\rho},\quad \frac{dm'}{ds}=-\frac{m}{\rho}. \tag{3·54}$$

It may be noted that we have nowhere assumed the continuity of $f''(x)$ or its equivalent. Formulae 3·51, 3·54 are the two-dimensional analogues of Frenet's Formulae.

A more general formula for ρ, available for any parametric representation of the curve, may be obtained as follows from 3·52:

$$\begin{aligned}\frac{(ds)^2}{\rho^2} &= (dl)^2+(dm)^2\\ &= \left(d\left(\frac{dx}{ds}\right)\right)^2+\left(d\left(\frac{dy}{ds}\right)\right)^2\\ &= \frac{(ds\,d^2x-dx\,d^2s)^2+(ds\,d^2y-dy\,d^2s)^2}{(ds)^4},\end{aligned}$$

which reduces to

$$\frac{1}{\rho^2}=\frac{(d^2x)^2+(d^2y)^2-(d^2s)^2}{(ds)^4}. \tag{3·55}$$

§ 3·60. **An alternative treatment using moving axes.** The theorems of § 3·50 can, alternatively, be obtained by the use of moving axes. This method has the advantage that it can be immediately generalized to provide a simple treatment of the corresponding problems for twisted curves.

The simplification obtained by this treatment is essentially due to the fact that the curve is at each point P referred to its tangent Pt and normal Pn as axes. Allowance must, of course, be made for the fact that, as the point P moves along the curve, the pair of axes Pt, Pn also move; in particular they rotate in their own plane; when P moves a distance δs along the curve the angle turned through is $\delta\psi$. If $d\psi/ds$ exists at P (and is equal to $1/\rho$) the axes are said to have the "spin" $1/\rho$ at P. For the sake of the kinematic analogy we suppose that P is moving along the curve with uniform unit velocity, so that s may be regarded as representing the time.

To obtain the results of § 3·50 we only require a knowledge of the true rates of increase of the components of a vector which is defined with respect to a system of moving axes. These may be stated in general terms as follows.

Let (x, y) be the vector components (functions of the time t) referred to axes moving with a spin ω, which is positive when it rotates the axis of x towards the axis of y. Let (x', y') be the differential coefficients of (x, y) with respect to t, and $(Dx/Dt, Dy/Dt)$ the true rates of change of the vector components in fixed directions coinciding with the instantaneous directions of the moving axes. Then*

$$Dx/Dt=x'-y\omega,\quad Dy/Dt=y'+x\omega. \tag{3·601}$$

Returning to our curve we see that any set of direction cosines defines a unit vector whose components may be regarded as functions of the "time" s. The direction cosines of the axes Ox, Oy referred to the moving axes Pt, Pn are respectively

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* The proof of this classical result is left to the reader.

$$(l, l'), (m, m').$$

These directions are fixed in space, and therefore (l, l'), (m, m') are the components of constant vectors. Therefore

$$\frac{Dl}{Ds}=\frac{Dl'}{Ds}=\frac{Dm}{Ds}=\frac{Dm'}{Ds}=0.$$

Applying 3·601, and replacing ω by $1/\rho$, we find that

$$(3\cdot602) \qquad \frac{dl}{ds}=\frac{l'}{\rho}, \quad \frac{dl'}{ds}=-\frac{l}{\rho}, \quad \frac{dm}{ds}=\frac{m'}{\rho}, \quad \frac{dm'}{ds}=-\frac{m}{\rho}.$$

But these are equations 3·51 and 3·54, which contain all the results of § 3·50.

§ 3·610. **The kinematics of a rigid body with reference to the properties of twisted curves.** Before leaving the subject of curvature, we propose to sketch shortly the manner in which the properties of a *twisted curve* may be obtained with the help of general geometrical theorems, usually applied to the problems of the kinematics of a rigid body. There is a special set of three lines at right angles through each point of a twisted curve (of which one is the tangent to the curve), referred to which the curve has a specially simple form near the point in question. These three straight lines or triad may be regarded as moving like a rigid body, as their point of intersection moves along the curve, and the nature of the curve is therefore closely connected with the motion of the triad.

The discussion forms a natural extension of the preceding part of the chapter, especially § 3·60. Proofs are, in the main, only outlined, and the reader may, with advantage, construct detailed proofs of any of the theorems enunciated.

It should be borne in mind in what follows that the kinematical language may be regarded simply as illustrative. All terms employed are capable of a purely geometrical or analytical interpretation.

In the first place, any displacement of a rigid body, with one point fixed, may be uniquely represented by a certain rotation about a certain axis through the fixed point, called the axis of rotation*.

Let us now regard the displacement of the body as defined by certain functions of an independent variable (the time t, say) which specify the direction cosines of certain lines fixed in the body referred to axes fixed in space. If, then, the differential coefficients of these functions exist at $t=t_0$, the axis of rotation tends to a limit called the **Instantaneous Axis** as $t \rightarrow t_0$, and the rotation about the axis is asymptotically equal to $\Omega\,(t-t_0)$, where Ω is a definite constant, which may be called the "spin" of the body about the instantaneous axis.

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* Euler's Theorem. See Routh, *Rigid Dynamics*, Vol. I, Chap. V, § 1; Thomson and Tait, *Natural Philosophy*, Vol. I, Part I, p. 69.

It follows that the motion of the body from t_0 to t can be represented by a rotation $\Omega(t-t_0)$ about the instantaneous axis, with an error $o\{|t-t_0|\}$*.

The "spin" obeys the vector laws of composition and resolution. It can be represented as a vector whose direction is the direction of the instantaneous axis and whose magnitude is Ω†. This may be established as follows.

If the body has the spin Ω about an axis whose direction cosines are (l, m, n), a point P in the body, of coordinates (x, y, z), has a velocity whose components are

$$\begin{vmatrix} m & n \\ y & z \end{vmatrix}\Omega, \quad \begin{vmatrix} n & l \\ z & x \end{vmatrix}\Omega, \quad \begin{vmatrix} l & m \\ x & y \end{vmatrix}\Omega, \qquad \text{(A)}$$

along the axes of Ox, Oy, Oz. If we agree to call $(l\Omega, m\Omega, n\Omega)$ the resolved parts of the spin Ω along these axes and write $\omega_1 = l\Omega$, etc., then the components of the velocity of P are

$$-\omega_3 y + \omega_2 z, \quad -\omega_1 z + \omega_3 x, \quad -\omega_2 x + \omega_1 y,$$

which are exactly the same components as are got by considering the body as having *simultaneously* spins ω_1, ω_2, ω_3 about the three axes Ox, Oy, Oz, respectively.

Therefore *spin can be resolved like a vector*, if the direction of this vector is the axis of the spin, and its length proportional to the spin.

In general a body may be said to have simultaneously two spins about any two different axes, if every point P of the body has a velocity which can be represented by the sum of two expressions of type (A). The Ox-component is then

$$-\omega_3 y + \omega_2 z - \omega_3' y + \omega_2' z,$$

or

$$-(\omega_3 + \omega_3') z + (\omega_2 + \omega_2') y.$$

Thus the body is really spinning about an axis and with a spin which is the resultant by the parallelogram law of the components

$$(\omega_1 + \omega_1'), \ (\omega_2 + \omega_2'), \ (\omega_3 + \omega_3'),$$

which is identical with the resultant by the same law of (*a*) $(\omega_1, \omega_2, \omega_3)$ and (*b*) $(\omega_1', \omega_2', \omega_3')$, *i.e.* of Ω and Ω'.

Therefore *spins may be compounded like vectors.*

The relation between the direction of rotation of the spin Ω and the positive direction of the axis denoted by (l, m, n) is chosen to be such that ω_1, the spin about Ox, is positive when it tends to rotate the axis Oy towards the axis Oz. The same statement is true of the other components when the letters (x, y, z) are cyclically interchanged.

Again suppose that a body, moving about a fixed point O, has an instantaneous axis at $t = t_0$. If two lines fixed in the body have positions OT_1, OT_2 at time t and OT_1', OT_2' at time t_0, and if

$$T_1\hat{O}T_2' = T_1\hat{O}T_2 + o\{|t-t_0|\}$$

the instantaneous axis must lie in the plane T_1OT_2‡.

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* Routh, *loc. cit.* Art. 217. The necessary extension of this article is obvious.

† Routh, *loc. cit.* Arts. 230–232.

‡ The reader should draw a spherical figure, and apply the preceding properties of the instantaneous axis.

This completes the properties of the spin and instantaneous axis that are required. We want one more property, the analogue for three dimensions of 3·601, concerning the true rates of increase of a vector defined with respect to a system of moving axes.

Let (x, y, z) be the vector components (functions of t) referred to moving axes whose motion may be resolved into spins $(\theta_1, \theta_2, \theta_3)$ about the instantaneous directions of these axes. Let (x', y', z') be the differential coefficients of (x, y, z) with respect to t, and

$$Dx/Dt,\ Dy/Dt,\ Dz/Dt$$

the true rates of change of the vector components in fixed directions coinciding with the instantaneous directions of moving axes. Then*

$$Dx/Dt = x' - y\theta_3 + z\theta_2,$$
$$Dy/Dt = y' - z\theta_1 + x\theta_3,$$
$$Dz/Dt = z' - x\theta_2 + y\theta_1.$$

This completes the account of the kinematics of a rigid body, so far as required for the discussion of a twisted curve.

§ 3·620. **The curvature and torsion of a twisted curve. Frenet's formulae.** The set of fundamental axes associated with a point P of a twisted curve are:—

Axis (1). **The tangent to the curve at** P, drawn in the direction of s increasing. This direction may be arbitrarily chosen for s.

Axis (2). **The principal normal at** P, which is the line through P, in the osculating plane, normal to the tangent at P. The positive direction of the principal normal may be chosen arbitrarily.

Axis (3). **The binormal at** P, which is the normal through P to the osculating plane, drawn in a direction such that the two triads, $P123$ and $Oxyz$, are of the same type, *i.e.* can be placed so that the positive directions $(P1, P2, P3)$, (Ox, Oy, Oz) respectively coincide.

Referred to axes $Oxyz$ let (l_1, m_1, n_1) be the direction cosines of the tangent $P1$, (l_2, m_2, n_2) the direction cosines of the Principal Normal $P2$, (l_3, m_3, n_3) the direction cosines of the Binormal $P3$.

The twisted curve is assumed to be given in the form

$$x = \phi_1(s),\quad y = \phi_2(s),\quad z = \phi_3(s),$$

and $\phi_1', \phi_2', \phi_3'$ are not simultaneously zero.

If $\phi_1''', \phi_2''', \phi_3'''$ exist, and s, for the sake of the kinematic analogy, is regarded as the "time", the triad $P123$, regarded as a rigid body, has an instantaneous axis, which lies in the plane $P13$, *i.e.* normal to $P2$†.

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* Routh, (*Advanced*) *Rigid Dynamics*, Vol. II, pp. 1–4.

† The directions of $P123$ may be defined in terms of $(\phi_1', \phi_2', \phi_3')$ and $(\phi_1'', \phi_2'', \phi_3'')$. Hence the instantaneous axis exists. By the theorem of Ex. 19 β, if $Q1'$ is the direction of the tangent at Q, and $PQ = \delta s$, the angle between $P3$ and $Q1'$ is $\frac{1}{2}\pi + O(\delta s)^2$, and the statement above follows from the last property of the Instantaneous Axis established in § 3·610.

The angular motion of the triad $P123$ may therefore be represented by spins $-1/\tau$, 0, $1/\rho$ about the axes $P1$, $P2$, $P3$. It therefore follows that, if P, Q are two points δs apart, $\delta\psi$ the angle between the tangents at P and Q, and $\delta\epsilon$ the angle between the osculating planes or binormals at P and Q, and if also $(\phi_1''', \phi_2''', \phi_3''')$ exist, then

$$\operatorname*{Lt}_{\delta s\longrightarrow 0}\frac{\delta\psi}{\delta s}=\frac{1}{\rho},\quad \operatorname*{Lt}_{\delta s\longrightarrow 0}\frac{\delta\epsilon}{\delta s}=-\frac{1}{\tau}.$$

We define $1/\rho$ and $1/\tau$ to be the **curvature** and **torsion** of the curve at P.

The direction cosines of the axes of reference Ox, Oy, Oz referred to the moving system $P123$ are respectively

$$(l_1,\, l_2,\, l_3),\ \ (m_1,\, m_2,\, m_3),\ \ (n_1,\, n_2,\, n_3).$$

These directions are fixed in space and therefore, for example, (l_1, l_2, l_3) are the components of a constant vector. Therefore

$$\frac{Dl_1}{Ds}=\frac{Dl_2}{Ds}=\ldots=\frac{Dm_1}{Ds}=\ldots=\frac{Dn_3}{Ds}=0.$$

It follows at once that

(3·621) $$0=\frac{dl_1}{ds}-\frac{l_2}{\rho},$$

(3·622) $$0=\frac{dl_2}{ds}+\frac{l_3}{\tau}+\frac{l_1}{\rho},$$

(3·623) $$0=\frac{dl_3}{ds}-\frac{l_2}{\tau},$$

with similar expressions for the m's and n's. These are **Frenet's Formulae.**

In conclusion we shall establish a few of the more important formulae for ρ and τ which follow from Frenet's formulae.

Since $\Sigma l_1^2=1$, $\Sigma l_1 l_2=0$, where Σ refers to summation over the letters (l, m, n), with similar formulae for other suffixes, we have at once

(3·624) $$\frac{1}{\rho^2}=\Sigma\left(\frac{dl_1}{ds}\right)^2=\sum_{x,y,z}\left(\frac{d^2x}{ds^2}\right)^2,$$

(3·625) $$\frac{1}{\tau^2}=\Sigma\left(\frac{dl_3}{ds}\right)^2,$$

(3·626) $$\frac{1}{\rho\tau}=\Sigma\frac{dl_1}{ds}\frac{dl_3}{ds}.$$

Again, since (l_3, m_3, n_3) are the direction cosines of the normal to the osculating plane, it follows from Ex. I9, that

$$\frac{l_3}{\begin{vmatrix} m_1 & n_1\\ m_1' & n_1'\end{vmatrix}}=\frac{m_3}{\begin{vmatrix} n_1 & l_1\\ n_1' & l_1'\end{vmatrix}}=\frac{n_3}{\begin{vmatrix} l_1 & m_1\\ l_1' & m_1'\end{vmatrix}}=\frac{\Sigma l_3^2}{\begin{vmatrix} l_3 & m_3 & n_3\\ l_1 & m_1 & n_1\\ l_1' & m_1' & n_1'\end{vmatrix}}=\frac{1}{\frac{1}{\rho}\begin{vmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}}=\rho,$$

for the triad $P123$ or (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) is of the same type as $Oxyz$. Dashes denote differentiation with respect to s. Therefore

$$\frac{1}{\rho\tau}=\Sigma l_1' l_3'=\Sigma l_1'\rho'\begin{vmatrix} m_1 & n_1 \\ m_1' & n_1' \end{vmatrix}+\rho\Sigma l_1'\begin{vmatrix} m_1 & n_1 \\ m_1'' & n_1'' \end{vmatrix}=\rho\begin{vmatrix} l_1' & m_1' & n_1' \\ l_1 & m_1 & n_1 \\ l_1'' & m_1'' & n_1'' \end{vmatrix}.$$

Hence

(3·627)
$$\frac{1}{\rho^2\tau}=-\begin{vmatrix} l_1 & m_1 & n_1 \\ l_1' & m_1' & n_1' \\ l_1'' & m_1'' & n_1'' \end{vmatrix}=-\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}.$$

§ 3·70. **Evolutes and involutes***. From here onwards we shall need to assume more than the existence of ρ, and shall therefore assume the existence and continuity of all differentials or derivates that are mentioned. It is easy to see that, for the curve $y=f(x)$, the continuity of $d\rho/ds$ is equivalent to the continuity of $f'''(x)$, and so on. Such details are here of no great interest or importance.

DEFINITIONS. **The evolute** *of a given curve is the locus of its centre of curvature. Any curve which has a given curve for evolute is called* **an involute** *of the given curve.*

Thus a given curve is an involute of its evolute, but we shall see it is only one of many.

We shall make a rule of using capital letters for points, etc. belonging to the evolute, X, Y, Ρ, S, Ψ, L, M, L', M' (if required) having the same meanings for the evolute that x, y, ρ, s, ψ, l, m, l', m' have for the original curve, and all letters referring to corresponding points. Now we have

$$X=x+l'\rho,\quad Y=y+m'\rho.$$

Therefore

(3·701)
$$\begin{cases} dX=dx+l'd\rho+\rho dl'=l'd\rho, \\ dY=dy+m'd\rho+\rho dm'=m'd\rho, \end{cases}$$

so that

(3·71)
$$(dS)^2=(d\rho)^2.$$

So long as $d\rho$ does not change sign, we can choose a suitable direction in which to measure S on the evolute, and have

(3·711)
$$dS=d\rho\,\dagger.$$

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* Picard, *Traité d'Analyse*, 2nd Ed., Vol. I, pp. 350 sqq. We shall in future refer to this book as Picard.

† This is of course an exact relation.

Now we have as usual

$$(3{\cdot}712) \qquad dX = L\,dS,\ dY = M\,dS,$$

and as a consequence of 3·701 and 3·712 we see that $L = l'$, $M = m'$. We have thus established the following properties.

THEOREM 3·72. *The evolute touches each normal at the centre of curvature.*

THEOREM 3·73. *The arc of the evolute, corresponding to an arc of the original curve for which* ρ *constantly increases or decreases, is equal to the difference of the radii of curvature touching its extremities.*

The last theorem follows at once by integrating 3·711; for we thus get $S = \rho + c$, where c is a constant.

It follows at once from 3·711 that every plane curve whose curvature is constant is a circle, for, since $(dS)^2 = 0$, the centre of curvature at one point is the centre of curvature at every point of the curve. Therefore every point of the curve lies on a fixed circle.

A formula for P is easily found. For we have

$$\mathrm{P} = (dS/d\Psi) = (d\rho/d\Psi),$$

and it is evident that $|d\Psi| = |d\psi|$. Hence, without regard to sign,

$$(3{\cdot}74) \qquad \mathrm{P} = \frac{d\rho}{ds}\frac{ds}{d\psi} = \rho\frac{d\rho}{ds}.$$

The converse of Theorem 3·72 is also true.

THEOREM 3·75. *A curve which at every point touches a normal to a given curve* Γ *is the evolute of* Γ *(or part of it).*

This is, properly speaking, a particular case of Theorem 5·32 on envelopes, but a direct proof is easy. For if a normal to Γ touches the curve at C, then C is the limit of the point of intersection of neighbouring normals, and therefore a centre of curvature of Γ. Hence all points on the curve are centres of curvature of Γ, which proves the theorem.

§ 3·80. THEOREM 3·81. *To a given curve there belong infinitely many involutes.*

Let $P(x, y)$ be a point on a given curve, and $Q(\xi, \eta)$ a point on the tangent at P such that $PQ = -(s+a)$, where a is any constant. Then

$$\xi = x - (s+a)\,l, \quad d\xi = -(s+a)\,dl,$$

$$\eta = y - (s+a)\,m, \quad d\eta = -(s+a)\,dm,$$

and therefore

$$d\xi : d\eta = l' : m',$$

so that the tangent at Q to the locus of Q is parallel to the normal at P; *i.e.* QP is the normal to the locus of Q at Q. Hence the given curve

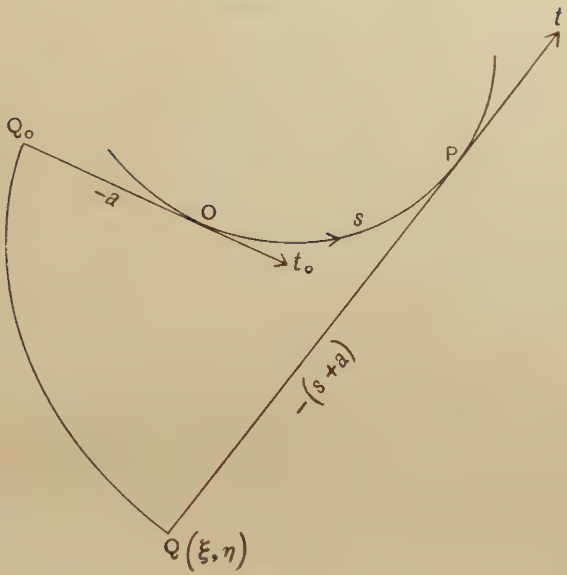

Fig. 11.

touches all the normals to the locus of Q, and each of its points is a point of contact with some normal. Therefore, by Theorem 3·75, the given curve is the evolute of the locus of Q, which is therefore an involute. It is clear that we get a new involute for each value of a.

The following mechanical description of the involute is of interest. Suppose that a thread, inextensible, perfectly flexible, and without thickness, is wound tightly round an arc of the curve, leaves the curve along the tangent at O, and ends on any chosen involute at Q_0. If the thread be then unwound off the curve, being always kept tight, the end will describe the chosen involute. For when the thread leaves the curve at P, and the end is at Q along the tangent at P, we have

$$QP - Q_0O = \text{arc } OP,$$

and therefore Q is still on the involute through Q_0. Hence the end describes the chosen involute.

EXAMPLES II

(1) Prove that, with due regard to sign,

$$\rho = r\,dr/dp$$

in all cases, p and r being the tangential-polar coordinates of the curve.

[Lamb, *Infinitesimal Calculus*, 2nd Ed., pp. 401–402.]

(2) By drawing the normals at points PQR such that arc $PQ=$ arc $QR=\delta s$, prove that the evolute touches the normals and that

$$dS=d\rho.$$

[Let $X_1 X_2$ be the mid-points of the arcs PQ, QR, and $C_1 C_2$ their centres of curvature; then C_1 is represented with an error $O(\delta s)^2$ by Γ_1, the intersection of normals at P and Q, and C_2 similarly by Γ_2, the intersection of normals at Q and R. Hence $\Gamma_1\Gamma_2 \sim C_1C_2$, and $\Gamma_1\Gamma_2$ tends to parallelism with the tangent to the evolute at C_1. Hence the theorem.]

(3) Taking the tangent and normal to a curve as the axes of x and y, and measuring s from the origin in the direction of x increasing, prove that

$$x=s-\frac{s^3}{3!}\left(\frac{d\psi}{ds}\right)_0^2+O(s^4),$$

$$y=\frac{s^2}{2!}\left(\frac{d\psi}{ds}\right)_0+\frac{s^3}{3!}\left(\frac{d^2\psi}{ds^2}\right)_0+O(s^4).$$

In particular if

$$\left(\frac{d\psi}{ds}\right)_0=\left(\frac{d^2\psi}{ds^2}\right)_0=\dots=\left(\frac{d^{r-1}\psi}{ds^{r-1}}\right)_0=0,$$

$(d^r\psi/ds^r)_0\neq 0$, and $d^{r+1}\psi/ds^{r+1}$ continuous, then

$$x=s-\frac{s^{2r+1}}{2r(4r^2-1)\{(r-1)!\}^2}\left(\frac{d^r\psi}{ds^r}\right)_0^2+O(s^{2r+2}),$$

$$y=\frac{s^{r+1}}{(r+1)!}\left(\frac{d^r\psi}{ds^r}\right)_0+O(s^{r+2}).$$

[We have $x=\int_0^s \cos\psi\, ds$, $y=\int_0^s \sin\psi\, ds$. Expand these expressions by Taylor's theorem, remembering that $\psi_0=0$.]

(4) If PT, QT are the tangents at P and Q, meeting in T, and the conditions of the latter part of (3) hold at P, then $PT/QT \to r$ as $Q \to P$.

[Use the result of Example (3) above.]

(5) If PT, QT are the tangents at P and Q, meeting in T, the circle PQT has, as its limit as $Q \to P$, a circle of radius $\frac{1}{2}\rho$ whose centre lies on the normal at P.

[If the normals at P and Q meet at K, $PQTK$ are concyclic.]

(6) If T_1, T_2, T_3 are the three points of intersection of the tangents at P, Q_1, Q_2, the circle $T_1T_2T_3$ has, as its limit as Q_1, $Q_2 \to P$, a circle of radius $\frac{1}{4}\rho$ whose centre lies on the normal at P.

[Use Theorem 2·624.]

CHAPTER IV

THE THEORY OF CONTACT*

§ 4·10. **The distance from a curve of a point near it.** If two curves have a common point P, it is a matter of some interest in itself, and of vital importance for further developments, to investigate how close the two curves lie to one another in the neighbourhood of P.

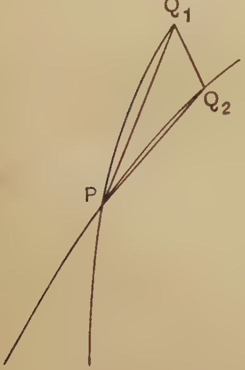

Fig. 12.

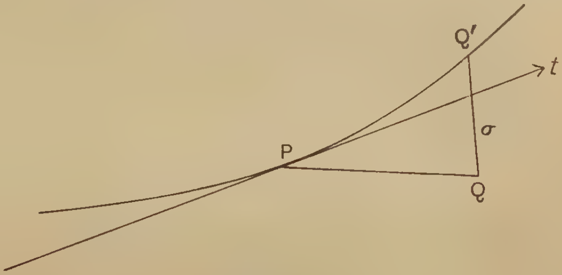

Fig. 13.

More precisely, it is required to determine the order of Q_1Q_2 (Fig. 12) when $Q_1 \to P$ and $Q_2 \to P$, *i.e.* when Q_1P and Q_2P are small lengths of the first order of smallness. With this object in view we proceed as follows.

We take the curve in the form

$$f(x, y) = 0,$$

and suppose that $P(a, b)$ is an *ordinary* point of the curve, *i.e.* that f_a' and f_b' are not both zero. Let Q be any point (x, y) near P, *not* on the curve, and Q' any point (x', y') near P on the curve, so that

$$f(a, b) = f(x', y') = 0.$$

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* d.l.V.P., Vol. II, p. 396 (on whose exposition our treatment is based); Picard, Vol. I, p. 342; Goursat, Vol. I, p. 530; Jordan, Vol. I, p. 417 (a treatment which includes complex points and curves).

Let the length and direction cosines of QQ' be σ and (l, m). We proceed to determine an asymptotic formula for σ valid as $Q \rightarrow P$. We assume that $f(x, y)$ has as many continuous differential coefficients as may be mentioned. At present our requirements are second order differential coefficients in the neighbourhood of P. We have

$$\begin{aligned} x' &= x + l\sigma, \quad y' = y + m\sigma, \\ 0 &= f(x + l\sigma, y + m\sigma), \\ &= f(x, y) + \sigma(lf_\xi' + mf_\eta')^*, \end{aligned}$$

where ξ is some number between x and x' and η some number between y and y'. Now when $Q \rightarrow P$ in any manner, l and m remaining constant and equal to l_0 and m_0, or satisfying the relations $l \rightarrow l_0$, $m \rightarrow m_0$, we have

$$f(x, y) \rightarrow f(a, b) = 0,$$

(4·101) $$lf_\xi' + mf_\eta' \rightarrow l_0 f_a' + m_0 f_b'.$$

If therefore $$l_0 f_a' + m_0 f_b' \neq 0,$$

i.e. if QQ' is parallel to, or tends to parallelism with, a fixed line L *not parallel to the tangent at P*†, then

(4·102) $$\sigma \rightarrow 0,$$

as $Q \rightarrow P$.

Again we have

$$\begin{aligned} 0 &= f(x, y) + \sigma(lf_x' + mf_y') + O(\sigma^2)‡, \\ &= f(x, y) + \sigma\{(l_0 f_a' + m_0 f_b') + o(1)\}, \end{aligned}$$

by 4·101 and 4·102, and, since

$$l_0 f_a' + m_0 f_b' \neq 0,$$

we have $$\sigma \sim \frac{-f(x, y)}{l_0 f_a' + m_0 f_b'},$$

as $Q \rightarrow P$. We can therefore enunciate the following theorem.

THEOREM 4·11. *If P is an ordinary point on the curve $f(x, y) = 0$, and σ is the distance from a neighbouring point $Q(x, y)$ to the curve, measured parallel§ to a straight line not parallel to the tangent at P, then*

$$\sigma \sim Af(x, y),$$

where $A \neq 0$‖, as $Q \rightarrow P$.

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* By Taylor's Theorem, for $n = 1$.

† This condition is sufficient (but not necessary) to ensure that $\sigma \rightarrow 0$.

‡ By Taylor's Theorem, for $n = 2$.

§ Or, '*in a direction that tends to parallelism with*'.

‖ Or, in the notation of *Orders of Infinity*, p. 2,

$$\sigma \asymp f(x, y).$$

The following alternative form may sometimes be more convenient.

THEOREM 4·12. *If* σ *is the shortest distance from* Q *to the curve, the other conditions of Theorem* 4·11 *remaining unaltered, then*

$$\sigma \sim -f(x, y)/(f_a'^2 + f_b'^2)^{\frac{1}{2}},$$

as $Q \to P$.

If QQ' is the shortest distance from Q to the curve, it is easily proved that QQ' is normal to the curve at Q'. As $Q \to P$, $\sigma \to 0$, and therefore $Q' \to P$, so that the direction parallel to which QQ' is measured tends to parallelism with the normal at P. Hence

$$l \to f_a'/(f_a'^2 + f_b'^2)^{\frac{1}{2}}, \quad m \to f_b'\,(f_a'^2 + f_b'^2)^{\frac{1}{2}},$$

and the theorem follows at once.

§ 4·20. **Definition of contact of order** n. We now make the following definition.

DEFINITION*. *Two curves that have an ordinary point* P *in common are said to have* **contact of order** n **at** P, *if the distance*† *of a point* Q *of one curve from the other is of the* $(n+1)$*th order of smallness compared to* QP.

They may be said to have **contact at least of order** n *if the distance is at least of the* $(n+1)$*th order of smallness compared to* QP.

Another way of stating the same thing is, of course, to say that if σ is the distance in question, then

$$\sigma \asymp (QP)^{n+1},$$

or in the second case

$$\sigma \preccurlyeq (QP)^{n+1}.$$

Suppose that the curve on which Q is not taken has the equation $f(x, y) = 0$. Then the results of the last section may be expressed as follows.

THEOREM 4·21. *The necessary and sufficient condition that the two curves should have contact of order* n *at the point* P *is that, when the coordinates of* Q *are substituted for* x *and* y *in* $f(x, y)$, *the expression* $f(x, y)$ *should satisfy the relation*

$$(4·211) \qquad f(x, y) \asymp (PQ)^{n+1},$$

as $Q \to P$ *along its curve.*

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[*] It should be observed that the definition as it stands needs justification, for it defines a property symmetrical with respect to the two curves in terms of an unsymmetrical property. But it will appear that the conditions of contact are symmetrical, so that Q may be taken on either curve, which affords the necessary justification.

[†] Distance may be taken to mean either (1) shortest distance, or (2) distance measured parallel to a line not parallel to the tangent of the other curve at P.

The following special cases are of the greatest importance.

I. Suppose that Q lies on the curve

$$x=\phi_1(t), \quad y=\phi_2(t),$$

that the other curve is

$$f(x, y)=0,$$

that P is the point of parameter t_0 on Q's curve, and that P is an ordinary point on both curves. Then, if Q is the point of parameter t,

$$PQ \asymp |t-t_0|,$$

as $Q \rightarrow P$ or $t \rightarrow t_0$. Following the general rule, it is necessary and sufficient for contact of order n that

$$f\{\phi_1(t), \phi_2(t)\} \asymp |t-t_0|^{n+1}.$$

Writing $$f\{\phi_1(t), \phi_2(t)\} \equiv \Phi(t),$$

and applying Taylor's Theorem, it appears that if $\Phi(t)$ has $n+1$ continuous differential coefficients at $t=t_0$, *the necessary and sufficient conditions for contact of order n are that*

$$(4{\cdot}212) \qquad \Phi(t_0)=\Phi'(t_0)=\ldots=\Phi^{(n)}(t_0)=0, \quad \Phi^{(n+1)}(t_0) \neq 0.$$

II. In particular suppose that the two curves are

$$y=f_1(x), \quad y=f_2(x).$$

If we take the latter for Q's curve, and $x=t$ as the parameter, then

$$\Phi(t) \equiv f_2(x)-f_1(x);$$

and, if the parameter of P is x_0, *the necessary and sufficient conditions are*

$$(4{\cdot}213) \qquad \begin{gathered} f_2(x_0)=f_1(x_0), \quad f_2'(x_0)=f_1'(x_0), \quad \ldots, \\ f_2^{(n)}(x_0)=f_1^{(n)}(x_0), \quad f_2^{(n+1)}(x_0) \neq f_1^{(n+1)}(x_0). \end{gathered}$$

It should be noticed that these conditions are symmetrical with respect to the two curves. Moreover this form of equation is perfectly general unless the tangent to the curves at P is parallel to the axis of y. But in this case the equations may be given in the form

$$x=f_1(y), \quad x=f_2(y),$$

leading, as before, to symmetrical conditions. It is therefore a matter of indifference on which curve Q is taken, and the symmetrical form of our original definition is justified. We note in passing the following theorems.

THEOREM 4·22. *Two curves which have contact of order n* (i.e. *of order n and of no higher order*) *at a point P cross or do not cross at P according as n is even or odd.*

THEOREM 4·23. *Two curves which have contact of order n with a third curve, have contact* **at least** *of order n with each other.*

The preceding treatment does not apply directly to the case of two curves whose equations are both given in parametric form. As this case seldom occurs in practice, and the preceding discussion is theoretically complete, we shall content ourselves here with a reference to Picard, Vol. I, pp. 342 sqq. The case of the two curves $F_1(x, y) = 0$, $F_2(x, y) = 0$ is considered by Jordan, *loc. cit.*, p. 420.

§ 4·30. **Osculating curves.** Given a curve and an ordinary point P thereon, and any family of curves depending on $n+1$ parameters, say

$$f(x, y, a_1, a_2, \dots, a_{n+1}) = 0, \tag{4·301}$$

suppose that a member T of this family is chosen by determining the parameters so that T has contact of the highest possible order for the family with the given curve at the point P. Then *T is said to have* **osculating contact** *with the given curve at the point P*: the two curves are also said *to osculate* or *to be osculating curves.*

Suppose that the given curve is

$$x = \phi_1(t), \quad y = \phi_2(t),$$

and that P is the point t_0. The conditions for contact of order n at the point P are by 4·212

$$\Phi(t_0) = \Phi'(t_0) = \dots = \Phi^{(n)}(t_0) = 0, \quad \Phi^{(n+1)}(t_0) \neq 0,$$

where

$$\Phi(t) \equiv f\{\phi_1(t), \phi_2(t), a_1, a_2, \dots, a_{n+1}\}.$$

In general these $n+1$ equations will be just sufficient to determine the $n+1$ parameters in the equation of the family (not necessarily uniquely); and *in general* it will not be the case that, when the a's have been so determined, $\Phi^{(n+1)}(t_0) = 0$. Therefore *in general the curve (or curves) of the family*

$$f(x, y, a_1, a_2, \dots, a_{n+1}) = 0 \tag{4·301}$$

that has (have) osculating contact with a given curve at the point P, has (have) contact of order n at that point.

The parameters of an osculating curve T are determined by $n+1$ equations which written in full are

$$\left\{\begin{array}{l} \Phi(t_0, a_1, a_2, \dots, a_{n+1}) = 0, \\ \Phi'(t_0, a_1, a_2, \dots, a_{n+1}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \Phi^{(n)}(t_0, a_1, a_2, \dots, a_{n+1}) = 0. \end{array}\right. \tag{4·302}$$

We shall denote a possible set of real values of the parameters, satisfying these equations, by $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$, and the corresponding osculating curve by T. Suppose now that K is a curve of the family which passes through P[*], and n neighbouring points on the given curve whose parameters are $t_0, t_1, t_2, \ldots, t_n$. The parameters ($a$'s) must satisfy the equations

$$(4{\cdot}303) \qquad \Phi(t_r, a_1, a_2, \ldots, a_{n+1}) = 0, \quad (r = 0, 1, 2, \ldots, n).$$

Without loss of generality we may suppose that $t_0 < t_1 < \ldots < t_n$. Then, by Rolle's Theorem, there are at least n distinct values of t, between t_0 and t_n, for which $\Phi'(t, a_1, a_2, \ldots, a_{n+1}) = 0$; by a second application of Rolle's Theorem there must therefore be at least $n - 1$ distinct values of t, between t_0 and t_n, for which $\Phi''(t_1, a_1, a_2, \ldots, a_{n+1}) = 0$, and so finally at least one value of t, between t_0 and t_n, such that

$$\Phi^{(n)}(t, a_1, a_2, \ldots, a_{n+1}) = 0.$$

The system of equations 4·303 may therefore be replaced by the equivalent system

$$(4{\cdot}304) \qquad \left\{\begin{array}{l} \Phi(t_0, a_1, a_2, \ldots, a_{n+1}) = 0, \\ \Phi'\{(t_0 + \mu_1), a_1, a_2, \ldots, a_{n+1}\} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi^{(n)}\{(t_0 + \mu_n), a_1, a_2, \ldots, a_{n+1}\} = 0, \end{array}\right.$$

where $\mu_1, \mu_2, \ldots, \mu_n \rightarrow 0$ as $t_1, t_2, \ldots, t_n \rightarrow t_0$, *i.e.* as

$$Q_1, Q_2, \ldots, Q_n \rightarrow P.$$

The equations 4·304 may be regarded as a system of $n + 1$ equations determining the a's as functions of the μ's. If the Jacobian[†] of this system does not vanish when

$$\mu_1 = \mu_2 = \ldots = \mu_n = 0, \quad a_1 = \alpha_1, \ \ldots, \ a_{n+1} = \alpha_{n+1},$$

that is *in general*, the system 4·304 determines a unique set of functions $\theta_1, \theta_2, \ldots, \theta_{n+1}$, such that

$$a_r = \alpha_r + \theta_r, \quad (r = 1, 2, \ldots, n + 1),$$

and θ_r is a function of $\mu_1, \mu_2, \ldots, \mu_n$, real and continuous when

$$|\mu_r| < \delta, \quad (r = 1, 2, \ldots, n),$$

and such that

$$\theta_r \rightarrow 0, \quad (r = 1, 2, \ldots, n),$$

when

$$\mu_1, \mu_2, \ldots, \mu_n \rightarrow 0.$$

---

[*] It is convenient to take this form of hypothesis, but not necessary. K may be taken to be a curve through $Q_1, Q_2, \ldots, Q_{n+1}$, $n + 1$ points near P, and $Q_1, Q_2, \ldots, Q_{n+1}$ made to tend to P.

[†] Goursat, Vol. I, Chap. III, in particular, pp. 96–97.

It follows that, if $Q_1, Q_2, \ldots, Q_n$ be sufficiently near P, a unique* curve K of the family can be drawn through $P, Q_1, Q_2, \ldots, Q_n$, which is such that, when $Q_1, O_2, \ldots, Q_n \to P$, K has the limit T.

In particular if the parameters appear in $f(x, y, a_1, \ldots, a_{n+1})$ in such a way that $a_1, a_2, \ldots, a_{n+1}$ are determined uniquely by the conditions for contact of order n, as is usually the case in practice, the curve K will be genuinely unique and tend to the unique limit T.

With the reservations that the foregoing analysis has disclosed, we can enunciate the following theorem.

THEOREM 4·31. *In general, the member of the family* 4·301, *which has osculating contact with a given curve at* P, *is a limit of curves of the family which pass through* P *and* n *neighbouring points on the given curve.*

This theorem enables us to attach a meaning to, and renders permissible, the use of such statements as "Two curves cut at P in n coincident points", statements which are formulated for the sake of generality, *e.g.* in order to enable us to say that any two conics have four common points real or complex. Their permissibility being established, such statements are frequently useful and illuminating.

§ 4·40. **Examples of osculating curves.** We have defined a tangent at P as the limit of the chord PQ when $Q \to P$. Since any straight line can be put in the form $\alpha x + \beta y - 1 = 0$, the osculating straight line at the point P is by the last theorem the tangent.

We can of course arrive at this fact directly as follows. Suppose the given curve is

$$x = \phi_1(t), \quad y = \phi_2(t):$$

then

$$\Phi(t) \equiv \alpha\phi_1(t) + \beta\phi_2(t) - 1;$$

and the conditions for first order contact at t_0 are

$$\alpha\phi_1(t_0) + \beta\phi_2(t_0) - 1 = 0, \quad \alpha\phi_1'(t_0) + \beta\phi_2'(t_0) = 0,$$

so that the osculating straight line is

$$\begin{vmatrix} x & y & 1 \\ \phi_1(t_0) & \phi_2(t_0) & 1 \\ \phi_1'(t_0) & \phi_2'(t_0) & 0 \end{vmatrix} = 0,$$

which is the tangent. It should be remembered that, t_0 being an ordinary point, $\phi_1'(t_0)$ and $\phi_2'(t_0)$ are not both zero.

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* It must not be supposed that K is necessarily the only curve of the system through $P, Q_1, \ldots Q_n$. K is the only curve of this nature whose parameters are "nearly equal to the a's".

We should also observe that *in general* the curve does not cross its tangent. If however it happens that the contact is of the second order at t_0, *i.e.* if also

$$\alpha\phi_1''(t_0)+\beta\phi_2''(t_0)=0,$$

or

$$\begin{vmatrix} \phi_1'(t_0) & \phi_2'(t_0) \\ \phi_1''(t_0) & \phi_2''(t_0) \end{vmatrix}=0,$$

so that t_0 is a point of inflexion, the curve will, in general, cross its tangent.

In §§ 3·10, 3·20 we defined the circle of curvature at a point $P(t_0)$ and proved that, if $\phi_1''(t)$ and $\phi_2''(t)$ are continuous at t_0, the circle of curvature is the limit of a circle passing through P and any neighbouring points Q_1 and Q_2, when $Q_1, Q_2 \rightarrow P$. The equation of any circle can be put in the form

$$(x-a)^2+(y-b)^2-R^2=0,$$

and a, b, R^2 are determined uniquely by the conditions of second order contact, except at a point of inflexion*.

It follows from the last theorem (as for a tangent) that the circle of curvature and the osculating circle are identical.

Or directly, for the curve $y=f(x)$, supposing that $f''(x) \neq 0$, we must have

$$(4·41) \qquad \begin{cases} (x-a)^2+\{f(x)-b\}^2-R^2=0, \\ x-a+f'(x)\{f(x)-b\}=0, \\ 1+f'^2(x)+f''(x)\{f(x)-b\}=0, \end{cases}$$

so that

$$f(x)-b=-\frac{1+f'^2(x)}{f''(x)},$$

$$x-a=\frac{f'(x)\{1+f'^2(x)\}}{f''(x)},$$

$$R^2=\{1+f'^2(x)\}^3/f''^2(x);$$

which agree with the equations of § 3·10. We notice that the circle of curvature has in general second order of contact, and therefore crosses the curve except at points at which, besides 4·41, the equation

$$3f''f'+f'''(f-b)=0$$

is satisfied. These are the points at which

$$(4·42) \qquad 3f''^2f'-(1+f'^2)f'''=0;$$

and it is easily verified that 4·42 is the same as $dR/ds=0$.

A direct corollary of Theorem 4·23 may be noticed, namely that two curves which have second order contact at any point have the same osculating circle at that point, and so the same radius of curvature.

§ 4·430. **Extension of the theory of osculating curves.** In § 4·30 we defined those members of a given family of curves which have osculating contact with a given curve at P, and showed that, in general, any such osculating curve is the limit of members of the family which have $n+1$ points of intersection with the given curve, when these $n+1$ points of intersection tend to P. This property may be extended as follows. Let

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* At a point of inflexion the tangent has second order contact and the curvature is zero. This may be conveniently expressed by saying that "the radius of curvature is infinite and that the circle of curvature is the tangent".

$$g(x, y, a_1, a_2, \ldots, a_{n-\lambda})=0 \tag{4·431}$$

be a family of curves, depending on $n-\lambda$ parameters, all of which curves have contact at least of order λ with a given curve $x=\phi_1(t)$, $y=\phi_2(t)$, at the point $P(t_0)$. Then, as before, we say that the osculating curve* of the family is that member of the family which has contact of the highest possible order with the given curve at P. This osculating curve* will, in general, have contact of order n and, if

$$\Phi(t)=g\{\phi_1(t), \phi_2(t), a_1, a_2, \ldots, a_{n-\lambda}\},$$

the conditions for contact of order n which determine the a's are

$$\Phi^{(\lambda+r)}(t_0)=0, \quad (r=1, 2, \ldots, n-\lambda). \tag{4·432}$$

It will be observed that

$$\Phi(t_0)=\Phi^{(r)}(t_0)=0, \quad (r=1, 2, \ldots, \lambda),$$

for all values of the parameters, since all members of the family have contact at least of order λ. We can now argue exactly as before in § 4·30 and arrive at the following theorem.

THEOREM 4·433. *In general, the member of the family* 4·431, *which has osculating contact with a given curve at P, has contact of order n with the given curve, and is the limit of curves of the family which* [*have contact of order* λ *at P and*] *pass through* $n-\lambda$ *points on the given curve in the neighbourhood of P.*

As an example, the foregoing theorem may be used to identify the Newtonian circle of curvature with the osculating circle. The Newtonian circle of curvature (§ 3·40) is the limit as $Q \rightarrow P$ of the circle touching a given curve at P and cutting it at Q. The foregoing discussion shows that this limit is a circle with second order contact with the given curve at P, that is, as such a circle is unique, the osculating circle.

§ 4·50. **Similar problems in three dimensions.** The treatment of the distance of a point Q from a surface

$$f(x, y, z)=0,$$

when Q is near an ordinary point P of the surface, is substantially the same as the treatment for plane curves. We obtain the following theorem.

THEOREM 4·51. *If P(a, b, c) is an ordinary point*† *on the surface*

$$f(x, y, z)=0,$$

and σ *is the distance from a neighbouring point* $Q(x, y, z)$ *to the surface,*

---

* Or, curves.

† That is, a point such that f_a', f_b', f_c' are not simultaneously zero.

measured parallel to a straight line not parallel to the tangent plane at P, then*

$$\sigma \sim Af(x, y, z),$$

where $A \neq 0$, *as* $Q \rightarrow P$.

Under the same conditions, if σ *is the shortest distance,*

$$\sigma \sim -f(x, y, z)/(f_a'^2 + f_b'^2 + f_c'^2)^{\frac{1}{2}},$$

as $Q \rightarrow P$.

On the other hand, a slight complication is introduced into the treatment of a twisted curve

$$f(x, y, z) = 0, \quad g(x, y, z) = 0$$

by the fact that there are *two* equations. Defining P, Q, Q', and (l, m, n) as in § 4·10, P and Q' lie on both the surfaces $f = 0$ and $g = 0$, and Q lies off at least one surface, say $f = 0$. It is moreover assumed that QQ' is not parallel to, and does not tend to parallelism with, the tangent to the twisted curve at P. If P is an ordinary point, and (u, v, w) are the direction cosines of the tangent to the curve at P, (u, v, w) satisfy and are determined by the equations

$$(4{\cdot}511) \qquad uf_a' + vf_b' + wf_c' = 0, \quad ug_a' + vg_b' + wg_c' = 0\dagger.$$

Since by hypothesis (l, m, n) or the limits of (l, m, n) do not satisfy the relations

$$u/l = v/m = w/n,$$

at least one of the expressions

$$lf_a' + mf_b' + nf_c', \quad lg_a' + mg_b' + ng_c'$$

is not zero, and does not tend to zero; in particular if Q lies on $g = 0$, but not on $f = 0$,

$$lf_a' + mf_b' + nf_c' \neq 0\dagger.$$

We then prove that $\sigma \rightarrow 0$, and then that

$$f(x, y, z) + \sigma\{lf_a' + mf_b' + nf_c' + o(1)\} = 0,$$
$$g(x, y, z) + \sigma\{lg_a' + mg_b' + ng_c' + o(1)\} = 0.$$

---

* Or, in a direction that tends to parallelism with.

† Since $f=0$ and $g=0$ do not touch at P. Points at which $f=0$ and $g=0$ touch cannot be regarded as ordinary points on the curve of intersection. The equations $f=0$ and $g=0$ may be regarded as a pair of simultaneous equations determining say x and y as functions of z. Such a determination requires that the Jacobian

$$\partial(f, g)/\partial(x, y)$$

should not vanish. At a point of contact of $f=0$ and $g=0$ all the three Jacobians

$$\partial(f, g)/\partial(x, y), \quad \partial(f, g)/\partial(y, z), \quad \partial(f, g)/\partial(z, x)$$

vanish and no pair of coordinates can be determined by the usual theorems as functions of the third.

We thus have two expressions for σ of which one at least gives an intelligible result.

If for example, $lg_a' + mg_b' + ng_c'$ is equal to or tends to 0, $g(x, y, z)$ will be of a higher order of smallness than $f(x, y, z)$ [*i.e.* $g/f \to 0$], and then $\sigma \sim Af(x, y, z)$ is correct and $\sigma \sim Ag(x, y, z)$ is false. We sum up in the following theorem.

THEOREM 4·52. *With the notation of Theorem* 4·51, *the distance* σ *from an ordinary point of the curve*

$$f(x, y, z) = 0, \quad g(x, y, z) = 0$$

satisfies whichever of the relations

$$\sigma \sim Af(x, y, z), \quad \sigma \sim A'g(x, y, z)$$

gives a greater value to σ (*i.e. the lower order of smallness*).

We can now extend our definition of contact of order n to any pair of curves plane or twisted or to a curve and a surface. *They are said to have contact of order* n *at a common point* P *if the shortest distance from a neighbouring point* Q *on one curve to the other curve or to the surface is of the* $(n+1)$*th order of smallness compared to* PQ.

§ 4·60. **Contact of a curve and a surface. Osculating surfaces.** After § 4·50 the following statements offer no difficulty.

THEOREM 4·61. *The necessary and sufficient condition that a curve and a surface* $f(x, y, z) = 0$ *should have contact of order* n *at a common point* P *is that, when the coordinates of a neighbouring point* Q *on the curve are substituted for* (x, y, z) *in* $f(x, y, z)$, $f(x, y, z)$ *should be of the* $(n+1)$*th order of smallness compared to* PQ.

If the curve is given by

$$x = \phi_1(t), \quad y = \phi_2(t), \quad z = \phi_3(t),$$

and $$\Phi(t) \equiv f\{\phi_1(t), \quad \phi_2(t), \quad \phi_3(t)\},$$

and P is the point t_0, the above condition reduces to

$$\Phi(t_0) = \Phi'(t_0) = \ldots = \Phi^{(n)}(t_0) = 0, \tag{4·611}$$

together with the inequality $\Phi^{(n+1)}(t_0) \neq 0$.

By taking $z = f(x, y)$ for the surface and $z = f_2(x)$, $y = f_1(x)$ for the curve, we can prove that *if a curve has contact of order* n (*and so of no higher order*) *with a surface, it crosses or does not cross the surface according as* n *is even or odd.*

Given a surface depending on $n+1$ parameters, we can define the osculating surface to a given curve at a given point, and prove that it *is in general the limit of surfaces of its type which intersect the given curve at* P *and* n *neighbouring points.*

§ 4·70. **Contact of two twisted curves. Osculating curves.** Suppose one of the curves is given in the form

$$f(x, y, z) = 0, \quad g(x, y, z) = 0,$$

and consider an ordinary point P. The reader can easily prove the following theorem.

THEOREM 4·71. *The necessary and sufficient conditions that two twisted curves, one of which is given by $f(x, y, z) = 0$, $g(x, y, z) = 0$, should have contact at least of order n at P* is that, when the coordinates of a point Q on the other curve are substituted for (x, y, z) in $f(x, y, z)$ and $g(x, y, z)$, both these expressions should be of the $(n+1)$th order of smallness at least compared to PQ.*

If the contact is of order n, one at least of $f(x, y, z)$ and $g(x, y, z)$ must be of the $(n+1)$th order of smallness exactly.

If the other curve is given in the form

$$x = \phi_1(t), \quad y = \phi_2(t), \quad z = \phi_3(t),$$

and

$$\Phi(t) \equiv f\{\phi_1(t), \phi_2(t), \phi_3(t)\},$$

$$\Psi(t) \equiv g\{\phi_1(t), \phi_2(t), \phi_3(t)\},$$

then these conditions reduce to

$$(4·711) \qquad \begin{cases} \Phi(t_0) = \Phi'(t_0) = \ldots = \Phi^{(n)}(t_0) = 0, \\ \Psi(t_0) = \Psi'(t_0) = \ldots = \Psi^{(n)}(t_0) = 0, \end{cases}$$

while one at least of

$$\Phi^{(n+1)}(t_0), \quad \Psi^{(n+1)}(t_0)$$

is different from zero. Therefore there are exactly $2n+2$ conditions to be satisfied in order that two twisted curves may have contact of order n.

By considering two curves whose equations are given in the form

$$y = f_1(x), \quad z = f_2(x),$$

$$y = g_1(x), \quad z = g_2(x),$$

we can show that the conditions of contact are symmetrical with respect to the two curves; for the conditions for contact at least of order n at x_0 are

$$(4·712) \begin{cases} f_1(x_0) = g_1(x_0), f_1'(x_0) = g_1'(x_0), \ldots, f_1^{(n)}(x_0) = g_1^{(n)}(x_0), \\ f_2(x_0) = g_2(x_0), f_2'(x_0) = g_2'(x_0), \ldots, f_2^{(n)}(x_0) = g_2^{(n)}(x_0). \end{cases}$$

These conditions may be stated thus†: The conditions for contact at least of order n at x_0 are that y, z, and their first n differential coefficients with respect to x should be the same at x_0 for both curves*.

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* P must of course be an ordinary point.

† There is an obvious case of exception. We must naturally suppose that the tangent to the curves is not perpendicular to the x-axis.

Given a twisted curve depending on $2n+2$ parameters, we can define the osculating curve to a given curve at a given point, and prove that it is *in general the limit of curves of its type which intersect the given curve at P and n neighbouring points.*

Given a twisted curve depending on $2n+1$ parameters, we can in general only satisfy the conditions for contact of order $n-1$. We then have a family of curves with contact of order $n-1$, but no curve with contact of order n.

EXAMPLES III

(1) The locus of the foci of parabolas which have second order contact at a given point of a given curve is a circle.

[From Ex. II 6 deduce that, if ρ is the radius of curvature at any point of a parabola, a circle of radius $\frac{1}{4}\rho$ touching the parabola at that point passes through the focus. All the parabolas and the given curve have the same circle of curvature. Hence deduce the theorem.]

(2) Find the locus of the centres of spheres having second order contact at a given point of a given curve.

[The locus is a line parallel to the binormal through the centre of curvature. Use the results of § 3·620 to identify this locus, choosing the fundamental triad as axes of reference.]

(3) If a surface S touches a plane P along a curve C, the tangent to C at any point has third order contact with S.

[If C is $y=f(x)$, $z=0$, the surface S in the neighbourhood of $z=0$ can be put in the form $z=F\{y-f(x)\}$, $F(0)=F'(0)=0$, and we may suppose that $f(0)=f'(0)=0$. The tangent to C at the origin is $x=t$, $y=0$, $z=0$, and

$$\Phi(t)=0-F\{0-f(t)\}=-\tfrac{1}{2}F''(0)\{f(t)\}^2+O(t^6).]$$

(4) At each point M of a surface S, and through each tangent line to the surface at M, passes one (and only one) circle which has third order contact with S. Show that if M is not an umbilic there are in general ten circles which have fourth order contact with S at M.

[Darboux, *Bulletin des Sciences Math.*, ser. 2, tome IV, p. 348.]

(5) Define the osculating plane (after chapter IV) to a given curve at a given point. Show that it is unique, has in general second order contact with the curve, and is crossed by the curve. Show that this definition is consistent with previous definitions, *e.g.* that of Ex. I 9.

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* For the case of two curves both given in parametric form, see Picard, Vol. I, p. 359.

An osculating plane having contact of a higher order is called *stationary*; find the condition for a stationary osculating plane, and show that if all the osculating planes are stationary the curve is a plane curve.

[d.l.V.P., vol. I, p. 335, vol. II, pp. 405 and 221.]

(6) Define the osculating line and the osculating circle to a twisted curve and prove that they are the tangent and circle of curvature respectively, having in general contact of the first and second orders with the given curve.

(7) If two curves have contact of order n at P, and Q, Q' are two points, one on each curve, such that $QQ' \asymp (PQ)^{n+1}$, show that $PQ/PQ' \to 1$ as $Q \to P$.

[Picard, vol. I, pp. 342, 359.]

(8) Find the conditions for contact of order n for the curves

$$r=f(\theta), \quad r=g(\theta),$$

and use these conditions to obtain the tangent and the circle of curvature at any point of the curve.

CHAPTER V

THE THEORY OF ENVELOPES*

§ 5·10. **The definition of the envelope of a family of plane curves.** Consider a family of curves depending on one parameter, *viz.*

$$f(x, y, \alpha)=0, \tag{5·101}$$

where $f(x, y, \alpha)$ has as many continuous differential coefficients with respect to x, y, and α as may be mentioned: usually the first two orders will be sufficient. We suppose further that any singular points on any curve (α)† that may exist, *i.e.* points (x, y) satisfying $f=f_x'=f_y'=0$, are isolated points. We proceed to investigate the way in which the curve (α) is placed with respect to a "neighbouring" curve of the system, *i.e.* the curve $(\alpha+\delta\alpha)$, where $\delta\alpha$ is sufficiently small.

Let M be a point (x, y) on the curve (α) which is an *ordinary point* of the curve, so that, for these values of x, y, and α, at least one of f_x', f_y' is not zero. It follows that a region of values of x, y, and α can be determined including M, and containing no singular point of any admissible member of the family*. The equation of the neighbouring curve is $f(x, y, \alpha + \delta\alpha) = 0$, which can be written in the form

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* d.l.V.P., Vol. II, p. 408.

† *I.e.* the curve $f(x, y, \alpha)=0$, for which the parameter has a particular value α.

$$(5{\cdot}102) \qquad f(x, y, \alpha) + \delta\alpha f_\alpha' + O(\delta\alpha)^2 = 0.$$

The shortest distance σ of M from this curve satisfies the relation†

$$\sigma = Af(x, y, \alpha + \delta\alpha)(1 + \epsilon),$$

where $A \neq 0$, (x, y) are the coordinates of M, and $\epsilon \rightarrow 0$ when $\sigma \rightarrow 0$. Since M is an ordinary point, it is easy to see that $\sigma \rightarrow 0$ when $\delta\alpha \rightarrow 0$, so that $\epsilon \rightarrow 0$ when $\delta\alpha \rightarrow 0$. Therefore, as $\delta\alpha \rightarrow 0$,

$$\sigma = \{Af_\alpha'\delta\alpha + O(\delta\alpha)^2\}\{1 + o(1)\}.$$

This distance will be of the second or higher order of smallness if and only if

$$f_\alpha' = 0.$$

An ordinary point on the curve (α) whose distance from the curve $(\alpha + \delta\alpha)$ is of the second order of smallness at least is called a **characteristic point of the curve** (α). These points are *ordinary* points at which $f_\alpha' = 0$. It may happen‡ that a curve (α) is entirely composed of characteristic points, but in general they will be isolated.

DEFINITION. **The envelope** *of the family*

$$f(x, y, \alpha) = 0$$

is the locus of its isolated characteristic points.

If there is an envelope, its points satisfy the equations

$$(5{\cdot}11) \qquad f = f_\alpha' = 0,$$

and the equation of the envelope must therefore be sought for by eliminating α between these two equations, or by solving them for x and y in terms of α and so obtaining the parametric representation

$$x = x(\alpha), \quad y = y(\alpha).$$

The complete result of such an elimination or solution is called **the α-discriminant of** f. We cannot, however, be certain that any curve contained in the α-discriminant forms part of the envelope. For instance, it may be possible to satisfy both equations 5·11 by a value of α independent of x and y, and then the corresponding curve (α) will be

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* *I.e.* there exists a δ such that if

$$|\xi - x| \leqslant \delta, \quad |\eta - y| \leqslant \delta, \quad |A - \alpha| \leqslant \delta,$$

it is never true that $f_\xi'(\xi, \eta, A) = f_\eta'(\xi, \eta, A) = 0$. In this region of values of (x, y), no curve of the family whose parameter satisfies $|A - \alpha| \leqslant \delta$ has a singular point.

† See Theorem 4·12.

‡ This and similar statements are illustrated by examples in § 5·20.

composed entirely of characteristic points. Such a curve will be included in the eliminant of 5·11, and so is part of the α-discriminant, but not part of the envelope. Again, suppose that the family contains a locus of singular points given by $x = x(\alpha)$, $y = y(\alpha)$, the point on this locus corresponding to α being a singular point on the curve (α). The coordinates (x, y) of these singular points satisfy the equations

$$f = 0, \quad f_x' = 0, \quad f_y' = 0.$$

They also satisfy the equation obtained by differentiating the first equation with respect to α, *viz.*

$$f_x' \frac{dx}{d\alpha} + f_y' \frac{dy}{d\alpha} + f_\alpha' = 0,$$

where $dx/d\alpha$ and $dy/d\alpha$ are determined from the equations of the singular point locus. Therefore at all points of this locus,

$$f_\alpha' = 0.$$

Hence any locus* of singular points of curves of the family will be part of the α-discriminant, and must be distinguished from the envelope. When, however, these two classes of curves have been identified and discarded, any remaining curve or curves obtained in the foregoing manner constitute the envelope of the system.

§ 5·20. **Examples.** (i) Consider the family

$$a^2 f + (2a+1) h = 0,$$

where $f=0$, $h=0$ are the equations of any distinct regular curves. The equations to be satisfied by a characteristic point are

$$a^2 f + (2a+1) h = 0,$$
$$af + h = 0,$$

which are satisfied by every point on the curves $a=0$, which curve is $h=0$, and $a=-1$, which curve is $h=f$. The eliminant is

$$h(h-f) = 0,$$

so that in this case there is no envelope.

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[*] In general no such locus exists. Singular points of members of the family are all the points determined by the equations

$$f = f_x' = f_y' = 0.$$

We have assumed that such points occur for isolated values of x and y for any given value of a. In general they will also occur only for isolated values of a, *i.e.* for isolated members of the family. It is easy to see by constructing examples that such isolated singular points *may or may not* lie on the a-discriminant.

(ii) Consider the family

$$f \equiv (y-a)^2 - x^3 = 0.$$

The curve (a) has a cusp at $(0, a)$ and therefore the $x=0$ is a locus of singular points. But

$$f_a' \equiv -2(y-a) = 0,$$

so the eliminant is

$$x^3 = 0,$$

which is the cusp locus, and therefore not an envelope. There is in fact no envelope.

§ 5·30. **Properties of the envelope.** Suppose that the curve (α) is not entirely composed of characteristic points. In this (the general) case, the equations $f = f_\alpha' = 0$ determine a number of isolated points (ξ, η) on the curve (α) which are in general characteristic, but may be singular points. Let (x, y) be any point of intersection of the curves (α) and $(\alpha + \delta\alpha)$; then x and y are determined by the equations

(5·301) $$f(x, y, \alpha) = 0, \quad f(x, y, \alpha + \delta\alpha) = 0.$$

By an application of Rolle's theorem, these may be reduced to the equations

(5·302) $$f(x, y, \alpha) = 0, \quad f_\alpha'(x, y, \alpha + \mu) = 0,$$

where

$$0 < |\mu| < |\delta\alpha|.$$

When $\mu = 0$ the solutions of equations 5·302 are the points (ξ, η) specified above. In order, therefore, to prove that the limit, as $\delta\alpha \to 0$, of any point of intersection (x, y) is a characteristic or singular point of $f=0$, it is only necessary to show that any solution of 5·302 determining x and y as functions of μ, is such that $x \to \xi$ and $y \to \eta$ as $\mu \to 0$. This necessary fact follows at once from implicit function theorems* if the Jacobian J of the system does not vanish, *i.e.* if

$$J = \begin{vmatrix} f_x' & f_{\alpha x}'' \\ f_y' & f_{\alpha y}'' \end{vmatrix} \neq 0,$$

when the variables (x, y, μ) take the system of values $(\xi, \eta, 0)$†. Even if, however, $J = 0$, it still remains true that $x \to \xi$ and $y \to \eta$ as $\mu \to 0$. Consider for example a characteristic point (ξ, η) at which $J = 0$. We may suppose without loss of generality that an arc of the curve $f = 0$ including this point can be put in the form $y = g(x)$. Substituting this value of y in f_α' we see that possible values of x must satisfy the equation

(5·303) $$h(x, \mu) \equiv f_\alpha'\{x, g(x), \alpha + \mu\} = 0,$$

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* Goursat, Vol. I, Chap. III, pp. 96, 97.

† The geometrical meaning of $J \neq 0$ is that (ξ, η) is not a singular point of $f=0$, or of $f_\alpha'=0$, and that the curves $f=0$ and $f_\alpha'=0$ do not touch at (ξ, η).

where $h(x, 0)$ vanishes for $x = \xi$. It may be verified that the condition $J = 0$ is equivalent to $h_x' = 0$. The point $(\xi, 0)$ may perhaps be a singular point on the (x, μ) curve*. But whether it is or not, $h(x, \mu)$ vanishes by hypothesis for values of μ in the neighbourhood of $\mu = 0$. We can, therefore, by the arguments of chapter VI, prove that there are one or more real branches of the (x, μ) curve in the neighbourhood of $(\xi, 0)$ such that

$$x = \xi + \theta_1(\mu),$$

where $\theta_1(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. The corresponding value of y takes the form

$$y = \eta + \theta_2(\mu),$$

where $\theta_2(\mu) \rightarrow 0$ as $\mu \rightarrow 0$, and $\theta_2(\mu)$ is uniquely determinate when $\theta_1(\mu)$ is known. The proposition follows as before.

We have so far excluded singular points of $f = 0$. An examination of Example (ii) above shows that the limit of a point of intersection may in fact be a singular point of $f = 0$. Since, however, singular point loci are excluded from the envelope, we need not discuss this case further, admitting that it actually occurs. We have proved therefore that *the limit of any point of intersection of neighbouring curves is an isolated characteristic (or perhaps singular) point.*

The converse of this proposition is not always true. We shall content ourselves with proving it with the help of the explicit assumptions that $J \neq 0$, and $f_{aa}'' \neq 0$†. We wish to prove that if (ξ, η) is such an isolated characteristic point on the curve $f = 0$, and (x, y) a neighbouring point on the curve, then a value of δa can be found such that

$$f(x, y, a + \delta a) = 0,$$

and $\delta a \rightarrow 0$ as $x \rightarrow \xi$ and $y \rightarrow \eta$ along $f(x, y, a) = 0$. Since $J \neq 0$, there exists a unique solution of the equations

$$f(x, y, a) = 0, \quad f_a'(x, y, a + \mu) = 0,$$

near (ξ, η), such that

$$x = \xi + \theta_1(\mu), \quad y = \eta + \theta_2(\mu),$$

where $\theta_1(\mu) \rightarrow 0$ and $\theta_2(\mu) \rightarrow 0$ as $\mu \rightarrow 0$. Since $f_{aa}'' \neq 0$, $\theta_1(\mu)$ and $\theta_2(\mu)$ are not identically zero. Given a point (x, y) near (ξ, η) on $f = 0$ we can therefore find a unique number μ_1, near 0, such that

$$f_a'(x, y, a + \mu_1) = 0.$$

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[*] The point will be a singular point when in addition

$$h_\mu' \equiv f_{aa}'' = 0.$$

[†] When $J = 0$ for all values of a, there may be an envelope in spite of the fact that neighbouring curves do not intersect. See § 5·50. We call points at which $J \neq 0$ and $f_{aa}'' \neq 0$ *completely ordinary points*. See § 5·310.

By an application of Taylor's theorem $f_\alpha'(x, y, \alpha+\mu)$ can be cast into the form

$$(5{\cdot}304) \qquad a(x-\xi)(1+\epsilon_1) - b\mu(1+\epsilon_2).$$

In 5·304, y has been eliminated by using $f=0$; a can only vanish with J and therefore $a \neq 0$; b can only vanish with $f_{\alpha\alpha}''$ and therefore $b \neq 0$; and $\epsilon_1, \epsilon_2 \rightarrow 0$ as $x \rightarrow \xi$ and $\mu \rightarrow 0$.

Now since $$f(x, y, \alpha) = 0,$$

$$f(x, y, \alpha+\mu_2) = \int_0^{\mu_2} f_\alpha'(x, y, \alpha+\mu)\, d\mu$$
$$= a\mu_2(x-\xi)(1+\epsilon_1) - \tfrac{1}{2}b\mu_2^2(1+\epsilon_2).$$

Since 5·304 must vanish for $\mu=\mu_1$, $a(x-\xi)$ and $b\mu$ will have the same sign, which may be supposed positive. It follows that, as μ_1 is unique, $f_\alpha'(x, y, \alpha+\mu)$ is positive when $0 \leqslant \mu < \mu_1$ and vanishes *and changes sign* at μ_1. Therefore $f(x, y, \alpha+\mu_1) > 0$, while by taking μ_2 sufficiently large we can ensure that $f(x, y, \alpha+\mu_2) < 0$. At the same time the requisite value of μ_2 can be made as small as we please by suitably diminishing $|x-\xi|$. There exists therefore a value of $\delta\alpha$, such that

$$f(x, y, \alpha+\delta\alpha) = 0, \quad (|\mu_1| < |\delta\alpha| < |\mu_2|);$$

moreover $\delta\alpha \rightarrow 0$ as $x \rightarrow \xi$, $y \rightarrow \eta$, and therefore the proposition is proved. We summarize this discussion in the following theorem.

THEOREM 5·31. *The locus of the limits of the intersections of neighbouring curves is in general the envelope, but may be a locus of singular points. Conversely, the envelope is in general* ($J \neq 0$, $f_{\alpha\alpha}'' \neq 0$) *the locus of the limits of these intersections, but there may be an envelope when neighbouring curves do not intersect.*

It should be observed that "the locus of the limits of the intersections of neighbouring curves which are not singular points" is not suitable for a definition of the envelope of the system, for, as is shown in § 5·50, on that definition a curve is not the envelope of its circles of curvature.

§ 5·310. **Properties of the envelope continued. Contact with members of the family.** We now proceed to consider the behaviour of the envelope in relation to members of the family at ordinary points of the envelope. It is necessary to start with a warning. It is natural for such arguments to take any arc of the envelope as expressed in the form

$$x = x(\alpha), \quad y = y(\alpha),$$

where α is the parameter of the point on the envelope corresponding to the curve (α). In order that we may do so without detailed investigations, it is necessary to apply the Implicit Function Theorem to the equations

$$f(x, y, \alpha)=0, \quad f_\alpha'(x, y, \alpha)=0,$$

so as to express x and y as functions of the parameter α.

This is possible as above* if

$$(5{\cdot}311) \qquad J=\begin{vmatrix} f_x' & f_{\alpha x}'' \\ f_y' & f_{\alpha y}'' \end{vmatrix} \neq 0$$

for the system of values, (x_0, y_0, α_0) say, in the neighbourhood of which we wish to discuss the behaviour of the envelope. The behaviour of the system in the neighbourhood of an isolated point at which $J=0$ is very interesting geometrically, and we shall return to this case later. When $J=0$, we cannot assert without further investigation that the envelope can be represented by $x=x(\alpha)$, $y=y(\alpha)$, where $x(\alpha)$ and $y(\alpha)$ have differential coefficients at $\alpha=\alpha_0$. In general we shall find that in such a case $x'(\alpha)$ and $y'(\alpha)$ tend to infinity as $\alpha \rightarrow \alpha_0$. We therefore assume for the present $J \neq 0$.

Let M be a point on the envelope, of coordinates (x, y) and parameter α, at which $J \neq 0$. It is therefore an ordinary point on the curve (α). Then in the neighbourhood of this point the envelope can be put in the form

$$x=x(\alpha), \quad y=y(\alpha),$$

where $x'(\alpha)$, $y'(\alpha)$ exist and are continuous.

The coordinates of M are functions of α which satisfy identically

$$f(x, y, \alpha)=0, \quad f_\alpha'(x, y, \alpha)=0.$$

On differentiating with respect to α, we find that they also satisfy

$$x'(\alpha)f_x' + y'(\alpha)f_y' + f_\alpha' = 0,$$

$$x'(\alpha)f_{\alpha x}'' + y'(\alpha)f_{\alpha y}'' + f_{\alpha\alpha}'' = 0.$$

Now $f_\alpha'=0$, and therefore, since $J \neq 0$, the necessary and sufficient condition that $x'(\alpha)$ and $y'(\alpha)$ should not be simultaneously zero is $f_{\alpha\alpha}'' \neq 0$.

Let us therefore suppose that M is a point on the envelope at which $J \neq 0$ and $f_{\alpha\alpha}'' \neq 0$. Such a point may be called a *completely ordinary point*; and in general all points will be such with the exception of isolated points. At M

$$x'(\alpha)f_x' + y'(\alpha)f_y' = 0,$$

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* Goursat, *loc. cit.*

but as neither both f_x' and f_y' are zero, nor both $x'(\alpha)$ and $y'(\alpha)$, it follows that the tangents to the curve and the envelope are parallel, *i.e.* identical. Therefore the envelope touches the curve (α) at M.

Conversely, let E be a curve which at every point* touches a member of the family. Then E is a locus of points of contact with members of the family, and therefore, assuming that in general only one member of the family touches E at a given point, the coordinates (x, y) of a point of E may be expressed as functions of α, $x(\alpha)$, $y(\alpha)$ say, which satisfy identically

$$f(x, y, \alpha) = 0.$$

By definition, the curve E has everywhere a tangent, but we cannot therefore assert that $x'(\alpha)$ and $y'(\alpha)$ must in general exist and be not both zero; and in fact no obvious method of proof on these lines presents itself. We therefore abandon the symmetry of the parametric representation and proceed as follows.

Let (x_0, y_0) be any point on E at which E is touched by the unique member of the family (α_0). Let (x, y) be a neighbouring point on E at which E is touched by (α). We have therefore

$$f(x_0, y_0, \alpha_0) = 0, \quad f(x, y, \alpha) = 0.$$

Now let $(x, y) \to (x_0, y_0)$. Since $f(x, y, \alpha)$ is a continuous function of the variables,

$$f(x, y, \alpha) = f(x_0, y_0, \alpha) + o(1).$$

Therefore, as $(x, y) \to (x_0, y_0)$,

$$f(x_0, y_0, \alpha) \to 0.$$

If α_0 is the only root of the equation $f(x_0, y_0, \alpha) = 0$, then, since $f(x_0, y_0, \alpha)$ is a continuous function of α, it follows that $\alpha \to \alpha_0$ as $(x, y) \to (x_0, y_0)$. In other words, neighbouring points on the curve E correspond to neighbouring values of α. This result however still remains true when the root α_0 is not unique, provided that, as we have assumed, the curve (α_0) is the only member of the family which touches E at (x_0, y_0).

Suppose for simplicity that there is only one other root α_1, so that $f(x_0, y_0, \alpha_1) = 0$, but the curve (α_1) does not touch E at (x_0, y_0). The argument can be extended at once to the case of any finite number of other roots, so that there is no loss of generality. In this case we can establish as before that, as $(x, y) \to (x_0, y_0)$, *either* (1) $\alpha \to \alpha_0$, *or* (2) $\alpha \to \alpha_1$, *or* (3) *there exists an infinite sequence of values of* α *for*

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* With the possible exception of isolated points.

which $\alpha \rightarrow \alpha_1$, *while the remaining values are such that* $\alpha \rightarrow \alpha_0$. Cases (2) and (3) may be ruled out by the following arguments. In the neighbourhood of (x_0, y_0), E is a curve having everywhere a tangent. Since this tangent is always also the tangent at (x, y) to the curve $f(x, y, \alpha)$, for values of α in the neighbourhood of α_0 and α_1, we may suppose without loss of generality that it is never parallel to the axis of y. Hence we may suppose that E is represented by an equation of the form $y = g(x)$, where $g'(x)$ exists at all points of the interval. Let m_0 and m_1 be the slopes of $f(x, y, \alpha_0) = 0$ and $f(x, y, \alpha_1) = 0$ at (x_0, y_0). Then $m_0 \neq m_1$. Moreover $g'(x_0) = m_0$ and, in cases (2) and (3), $g'(x)$ assumes a series of values tending to m_1 for values of x in the neighbourhood of x_0. This however is impossible by a theorem due to Darboux*, which states that, *if* $f(x)$ *has a differential coefficient at all points of an interval* (a, b), $f'(x)$ *cannot pass from one value to another in this interval without assuming every intermediate value.* It follows that $\alpha \rightarrow \alpha_0$ in all cases, which is what we required to prove.

Now consider the points (x, y, α) and $(x + \delta x, y + \delta y, \alpha + \delta\alpha)$ on E. Then E has a tangent at (x, y) which may without loss of generality be supposed not parallel to the axis of y.

Since $f(x, y, \alpha) = 0$, $f(x + \delta x, y + \delta y, \alpha + \delta\alpha) = 0$, we have

$$\delta x f_x'(\theta) + \delta y f_y'(\theta) + \delta\alpha f_\alpha'(\theta) = 0,$$

where $f_x'(\theta), \ldots,$ denote $f_x'(x + \theta\delta x, y + \theta\delta y, \alpha + \theta\delta\alpha), \ldots,$ and $0 < \theta < 1$. Now let $\delta x, \delta y \rightarrow 0$ along E. We have

$$f_x'(\theta) + \frac{\delta y}{\delta x} f_y'(\theta) = -\frac{\delta\alpha}{\delta x} f_\alpha'(\theta).$$

But
$$f_x'(\theta) + \frac{\delta y}{\delta x} f_y'(\theta) \rightarrow f_x' + y' f_y',$$

where y' refers to the curve E. The limit y' exists because E has a tangent at (x, y). Since E touches $f(x, y, \alpha) = 0$ at (x, y), $f_x' + y' f_y' = 0$. Also $f_\alpha'(\theta) \rightarrow f_\alpha'$. Therefore either $f_\alpha' = 0$, or $\delta\alpha/\delta x \rightarrow d\alpha/dx = 0$. The latter alternative may happen at isolated points, but cannot happen everywhere in an interval unless the curve E is identical with a member of the family. Hence in general $f_\alpha' = 0$.

If follows that E is a locus of isolated characteristic points or of singular points, and this latter case may actually occur†. We have therefore proved the following theorem.

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* d.l.V.P., Vol. I, p. 97.

† The family of curves $y^2 = (x - \alpha)^3$ have cusps at $(\alpha, 0)$. The line $y = 0$ touches all members of the family and touches one and only one at every point, and is also the cusp locus.

THEOREM 5·312. *The envelope touches the curve* (α) *at all its isolated characteristic points that are completely ordinary points of the envelope.*

Conversely, a curve which touches just one member of the family at every point is in general the envelope (or a part of it), but may be a locus of singular points.

This theorem supplies a possible alternative definition of the envelope, as "the most complete curve that touches a member of the family at every point, and which does not contain a locus of singular points". The theory is developed from this point of view by Goursat (Vol. I, p. 511); owing however to the difficulty of avoiding *a priori* assumptions as to the nature of $x'(a)$ and $y'(a)$, in the proof that on the envelope $f_a'=0$, this definition does not appear to be so suitable as the one chosen here.

A corollary of this last theorem is that any curve is the envelope of its tangents. A direct proof is not without interest. Let $y=f(x)$ be the given curve, so that the tangent at the point a is

$$y-f(a)-(x-a)f'(a)=0,$$

and the coordinates of the characteristic points satisfy

$$(x-a)f''(a)=0.$$

If $f''(a)=0$, every point on the tangent, which is then inflexional, is a characteristic point: discarding this case, the isolated characteristic point of the tangent at a is the point $x=a$, $y=f(a)$, and the locus of these points is the given curve.

§ 5·40. **Order of contact of the envelope and the curves. Meaning of $J=0$.** We have already proved that, at any completely ordinary point a of the envelope, the envelope touches the curve (a). It is easily seen that the contact is necessarily first order contact at such a point, for $J=0$ is a necessary condition for contact of higher order than the first*.

We now consider a point a on the envelope which is an ordinary point of the curve (a), but at which $J=0$; in general $f_{aa}''\neq 0$ at such a point; we shall suppose that this is the case. Near such a point we cannot assume at once that the parametric representation of the envelope $x=x(a)$, $y=y(a)$ is possible, where $x(a)$ and $y(a)$ possess differential coefficients. We can prove however by differentiating along the envelope, exactly as in the latter part of Theorem 5·312, *that $a'=da/dx$ exists at this point and, because $J=0$, is in fact zero, and that the envelope touches the curve* (a).

One aspect of the geometrical meaning of the conditions $J=0$, $f_{aa}''\neq 0$ at an isolated point of the envelope is therefore that *the distribution of points of contact of members of the family along the envelope is exceptionally sparse near such a point.*

We can however go further than this. We can prove step by step, with suitable assumptions as to the nature of $f(x, y, a)$, that a'', a''', ... exist when we differentiate along the envelope. Now the envelope satisfies identically $f(x, y, a)=0$ and $f_a'(x, y, a)=0$, and therefore satisfies identically

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* A proof is sketched in Ex. IV, 3.

$$f_x'+y'f_y'+a'f_a'=0, \tag{5·401}$$

$$f_{ax}''+y'f_{ay}''+a'f_{aa}''=0. \tag{5·402}$$

Differentiating 5·401 again along the envelope, and using 5·402 to simplify the result, we obtain

$$f_{xx}''+2y'f_{xy}''+y'^2f_{yy}''+y''f_y'-a'^2f_{aa}''+a''f_a'=0. \tag{5·403}$$

Differentiating 5·403, and using the facts that $a'=0$ at the point a, and that, as always, $f_a'=0$, we obtain

$$f_{xxx}'''+3y'f_{xxy}'''+3y'^2f_{xyy}'''+y'^3f_{yyy}''' \\ +3y''f_{xy}''+3y'y''f_{yy}''+y'''f_y'=0. \tag{5·404}$$

Equation 5·403 itself reduces to

$$f_{xx}''+2y'f_{xy}''+y'^2f_{yy}''+y''f_y'=0. \tag{5·405}$$

But 5·404 and 5·405 are precisely the expressions we obtain when we determine y'' and y''' for the curve (a) by differentiating $f(x, y, a)=0$ with a constant. It is easily verified that in general y^{iv} is different for the curve and the envelope. At this point therefore the curve (a) and the envelope have in general contact of the third order, i.e. *two orders higher than normal.* We can collect these results into the following theorem.

THEOREM 5·41. *At isolated points a on the envelope at which $J=0$, $f_{aa}''\neq 0$, and the curve (a) has an ordinary point,*

(i) *the envelope has in general third order contact with the curve (a),*

(ii) *$da/dx=da/dy=0$*, so that the distribution of points of contact of members of the family along the envelope is exceptionally sparse*†.

The condition $f_{aa}''=0$, $J\neq 0$, is in general satisfied at isolated points of the envelope. On referring to § 5·310, we see that this condition implies that $x'(a)=y'(a)=0$, so that the point is a singular point on the envelope. It is in fact in general a cusp of the first species (see Chapter VI).

If both the conditions $J=0$ and $f_{aa}''=0$ are satisfied at an isolated point of the envelope, the state of affairs is more complicated. In general the envelope has two branches through the point, both of which have second order contact with the curve (a). In certain cases one of these branches may coincide with the curve (a)‡. It is then no longer part of the envelope, but still remains part of the a-discriminant. For the further study of these and other singularities of the envelope, or more generally of the a-discriminant, the reader should refer to Bromwich and Hudson, *loc. cit.*

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* Assuming that the tangent to the curve or envelope is parallel to neither axis of coordinates.

† In this connection the reader should refer to a paper by Bromwich and Hudson (*Quarterly Journal*, Vol. XXXIII, p. 98) called "The discriminant of a family of curves or surfaces."

‡ An inflexional tangent is the simplest example. The reader should verify that $J=0$ and $f_{aa}''=0$ in this case. See p. 67.

§ 5·420. **Envelopes with contact everywhere of high order.** We have so far considered the behaviour of the envelope at isolated exceptional points. It may happen however that every point is exceptional, so that $J=0$ for all values of a. By definition, no arc of the envelope can coincide with any member of the family, and therefore a' can only vanish at isolated points. We have as usual, differentiating along the envelope,

$$f_x' + y' f_y' = 0,$$
$$f_{ax}'' + y' f_{ay}'' + a' f_{aa}'' = 0,$$

so that $J=0$ *for all values of* a *if and only if* $f_{aa}''=0$ *for all values of* a.

In general $f=0$ and $f_{aa}''=0$ will determine (by the implicit function theorem) a parametric representation of the envelope, $x=x(a)$, $y=y(a)$, possessing differential coefficients near any value of a except those isolated values at which

(5·421)
$$J_1 = \begin{vmatrix} f_x' & f_{aax}''' \\ f_y' & f_{aay}''' \end{vmatrix} = 0.$$

When $J_1 \neq 0$, the point is an ordinary point on the envelope unless $f_{aaa}'''=0$.

The conditions for contact of order n between the envelope and the curve (a) are, by 4·212,

(5·422)
$$\left[\frac{\partial^r \Phi(t, a)}{\partial t^r}\right]_{t=a} = 0, \ (r = 1, 2, \ldots n),$$

(5·423)
$$\left[\frac{\partial^{n+1} \Phi(t, a)}{\partial t^{n+1}}\right]_{t=a} \neq 0,$$

where

$$\Phi(t, a) = f(x(t), y(t), a).$$

We observe that the equations

(5·424)
$$\Phi(t, t) = 0, \quad \Phi_a'(t, t) = 0, \quad \Phi_{aa}''(t, t) = 0$$

are satisfied for all values of t, where $\Phi_a'(t, t)$ denotes

$$[f_a'(x(t), y(t), a)]_{a=t}.$$

Making use of the identities 5·424 and the similar identities obtained by differentiation, we verify that 5·422 and 5·423 are satisfied for $n=2$. The envelope therefore has in general second order contact with all members of the family. At isolated points at which $J_1=0$, $f_{aaa}''' \neq 0$, we can show as in the last section that $a'=0$ and the envelope has in general third order contact with the curve (a). In this case the contact is only one order higher than usual.

Conversely, if the contact is in general second order with all members of the family, then $J=0$ and $f_{\alpha\alpha}''=0$ for all values of α, for otherwise contact is in general of the first order only. It therefore follows that the necessary and sufficient conditions that the envelope should in general have contact of the second order with all members of the family are that

$$f_{\alpha\alpha}''=0$$

for all values of α, and

$$f_{\alpha\alpha\alpha}'''\neq 0$$

except for isolated values of α.

The same arguments can be extended step by step to prove that the necessary and sufficient conditions that the envelope should in general have contact of order n with all members of the family are that

$$\frac{\partial^2 f}{\partial \alpha^2}=\frac{\partial^3 f}{\partial \alpha^3}=\ldots=\frac{\partial^n f}{\partial \alpha^n}=0 \tag{5·425}$$

for all values of α, and

$$\frac{\partial^{n+1} f}{\partial \alpha^{n+1}}\neq 0 \tag{5·426}$$

except for isolated values of α.

It should be observed that these conditions 5·425 and 5·426 bear no obvious relation whatever to the necessary and sufficient conditions for contact of order n at an isolated point of an envelope for which contact is in general of order $n-r$, where $r \geqslant 1$*.

In accordance with Theorem 4·22, the members of the family in general cross the envelope at their points of contact when n, the order of the contact, is even, and do not cross the envelope when n is odd. It can also be shown that in a family for which n is in general odd, neighbouring curves must intersect, while in a family for which n is in general even neighbouring curves do not intersect†. To prove these assertions an extension of the analysis of § 5·30 is required to the more complicated cases in which $J=0$. We shall content ourselves here with proving that, when $n=2$, neighbouring curves do not intersect. In this case we have $J=0$, $f_{\alpha\alpha}''=0$ everywhere and in general $f_{\alpha\alpha\alpha}'''\neq 0$.

The proof of § 5·30 that, when $n=1$, neighbouring curves must intersect depends essentially on the fact that, if (x, y) is any given point on the curve (α) near a characteristic point, a value μ_1 of μ can be found, near 0, such that

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* It is stated by Goursat (Vol. I, p. 549) in an example, that if 5·425 and 5·426 are satisfied for an *isolated* value α, they are the conditions for a contact of order n between the envelope and the curve (α). This is incorrect.

† This is geometrically obvious, or almost so, as can be seen by drawing a figure.

$$f_\alpha'(x, y, \alpha + \mu_1)$$

vanishes and *changes sign* when μ passes through the value μ_1. The failure of the argument for $n = 2$ (and in general when n is even) is due to the fact that the lowest order terms in $f_\alpha'(x, y, \alpha + \mu)$ (except for a constant factor) form a perfect square, and no value of μ exists for which f_α' changes sign near $\mu = 0$. Taking for simplicity the point (0, 0) as the characteristic point under discussion on the curve (α), it is easily verified that the lowest order terms in the expansion of $f_\alpha'(x, y, \alpha + \mu)$, near (0, 0) and near $\mu = 0$, are second order terms in x and μ, which can be put in the form

$$\frac{1}{2!}\{x^2(f_{\alpha xx}''' + 2y' f_{\alpha xy}''' + y'^2 f_{\alpha yy}''' + y'' f_{\alpha y}'')_0$$
$$+ 2x\mu(f_{\alpha\alpha x}''' + y' f_{\alpha\alpha y}''')_0 + \mu^2(f_{\alpha\alpha\alpha}''')_0\},$$

where y' and y'' refer to the curve (α). But both y' and y'' in this case have the same values for the envelope at (0, 0); also at all points of the envelope $f_\alpha' = f_{\alpha\alpha}'' = 0$. By differentiating these equations along the envelope we obtain finally

$$(f_{\alpha xx}''' + 2y' f_{\alpha xy}''' + y'^2 f_{\alpha yy}''' + y'' f_{\alpha y}'')_0 = \alpha_0'^2(f_{\alpha\alpha\alpha}''')_0,$$
$$(f'''_{\alpha\alpha x} + y' f'''_{\alpha\alpha y})_0 = -\alpha_0'(f'''_{\alpha\alpha\alpha})_0,$$

where $\alpha_0' = (d\alpha/dx)_0$, taken along the envelope ; in general α_0' is not zero. The second order terms therefore reduce to

$$\frac{1}{2!}(f_{\alpha\alpha\alpha}''')_0(x\alpha_0' - \mu)^2.$$

It follows that, for any given values of x and y on the curve (α) near (0, 0), f_α' cannot change sign for any value μ such that $|\mu| < A$, where A is a constant independent of x and y. For such values of μ, $\int_0^\mu f_\alpha' \, d\mu$ can never vanish. No value therefore of $\delta\alpha$ can be found for which $f(x, y, \alpha + \delta\alpha) = 0$, while $\delta\alpha$ tends to zero as (x, y) tends to a characteristic point on (α). Therefore no characteristic point is the limit of points of intersection of neighbouring curves. If such curves intersect at all they can only do so near a singular point. These results may be summarized thus:

THEOREM 5·427. *Under the conditions 5·425 and 5·426, the contact between the curve and its envelope is of order n, except at isolated points at which the order may be higher. Neighbouring curves (α) in general intersect and do not cross the envelope when n is odd, and do not intersect but cross the envelope when n is even.*

§ 5·50. **The envelope of a system of circles.** We will now give an example of a family for which neighbouring curves do not intersect, but which still possesses an envelope. Such a family is formed by the circles of curvature of any plane curve. For since the plane curve touches each circle at an ordinary point, it is the envelope (or part of it). Moreover, on referring to § 3·70, we see that the difference between the radii of curvature at neighbouring points is equal to the arc of the evolute between the two centres of curvature, and is therefore in general greater than the distance between these centres. One circle therefore completely encloses the other, and so there are no points of intersection.

It follows from § 5·420 that contact between members of the family and the envelope must in such a case be of even order, and therefore the typical case of an envelope not generated by limits of points of intersection is that in which this contact is in general of the second order. A family of circles is the simplest possible family of the kind, for the three conditions of second order contact require just three arbitrary functions of a in the equation of the curves of the family.

It is of interest to study a family of circles directly. We shall, among others, arrive at the foregoing results, and also find that a given curve is the complete envelope of its circles of curvature, a point at present in doubt. Let the system be

$$(x-a)^2+(y-b)^2-R^2=0 \tag{5·501}$$

where a, b, R are functions of a. There are no singular points on any member. The characteristic points are the intersections of this circle with the straight line

$$(x-a)\,a'+(y-b)\,b'+RR'=0. \tag{5·502}$$

Denoting the locus of the centres of the circles by C, we see that this line is perpendicular to the tangent to C at a, *i.e.* the point $a(a)$, $b(a)$, and distant from this point by

$$RR'/(a'^2+b'^2)^{\frac{1}{2}}\,*.$$

This is greater than, equal to, or less than R according as R'^2 is greater than, equal to, or less than $a'^2+b'^2$, and there are no, one, or two characteristic points respectively. In the first case there is no envelope for that part of the family, and, if $R'^2>a'^2+b'^2$ for all values of a, no envelope at all. Next suppose that for all values of a, $R'^2<a'^2+b'^2$. There is then an envelope, composed in general of two branches; an obvious example is provided by the case of a moving circle of constant radius $R'=0$. Finally suppose that for all values of a

$$R'^2=a'^2+b'^2. \tag{5·51}$$

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* Supposing a' and b' not both zero. If they both are, and $R'\neq 0$, there is no characteristic point, but if also $R'=0$, this particular circle is entirely composed of characteristic points. This can only happen for isolated values of a, which may be neglected.

There is then an envelope composed of the points of contact of 5·501 and 5·502. It follows that $(x-a)a'+(y-b)b'+RR'=0$ is the tangent to the envelope at the characteristic point of α, and therefore that normals to the envelope are tangents to C. In other words C is the evolute of the envelope, and therefore *the circles are the circles of curvature of their envelope.* It is easily seen that the condition 5·51 is also necessary for this relationship. Any plane curve is therefore the complete envelope of its circles of curvature.

§ 5·60. **Other rules for envelopes.** (1) The equation of the family may be given in some other form such as

$$f(x, y, \alpha, \beta) = 0 \tag{5·601}$$

with the condition

$$\phi(\alpha, \beta) = 0. \tag{5·602}$$

We apply the usual rule, regarding β as a function of α, and must therefore have

$$f_\alpha' + \frac{d\beta}{d\alpha} f_\beta' = 0. \tag{5·603}$$

But we also have

$$\phi_\alpha' + \frac{d\beta}{d\alpha} \phi_\beta' = 0 \tag{5·604}$$

and we have therefore to eliminate α, β, $\frac{d\beta}{d\alpha}$ between the equations (5·601—5·604), or, what is the same thing, α and β between

$$f = 0, \quad \phi = 0, \quad \frac{\partial(f, \phi)}{\partial(\alpha, \beta)} = 0. \tag{5·61}$$

All the former exceptional cases must be taken account of, with the addition of singularities of $\phi(\alpha, \beta) = 0$, but these, being isolated points, are not of importance.

(2) The curves may be given in the parametric form

$$x = \phi_1(t, \alpha), \quad y = \phi_2(t, \alpha). \tag{5·611}$$

We can apply the usual rule, regarding t as a function of y and α. We have therefore

$$\frac{\partial \phi_1}{\partial t}\frac{\partial t}{\partial \alpha} + \frac{\partial \phi_1}{\partial \alpha} = 0,$$

corresponding to $f_\alpha' = 0$, and also the identical relation

$$\frac{\partial \phi_2}{\partial t}\frac{\partial t}{\partial \alpha} + \frac{\partial \phi_2}{\partial \alpha} = 0,$$

which lead together to

$$\frac{\partial(\phi_1, \phi_2)}{\partial(t, \alpha)} = 0.$$

The envelope is therefore to be looked for, with the usual precautions, among the results of eliminating t and a from

$$(5{\cdot}62) \qquad x=\phi_1(t, a), \quad y=\phi_2(t, a), \quad \frac{\partial(\phi_1, \phi_2)}{\partial(t, a)}=0.$$

EXAMPLES IV

(1) *The envelope of the straight line*

$$(1) \qquad x\cos a+y\sin a=f(a).$$

The characteristic point lies on the straight line

$$(2) \qquad -x\sin a+y\cos a=f'(a).$$

Show that (i) the line (2) is normal to the envelope, (ii) the envelope of the line (2) is the evolute of that of the line (1), and (iii) that the curvature and arc of the envelope of the line (1) are given by

$$\rho=ds/da=\pm\{f(a)+f''(a)\}.$$

(2) *Families of circles, such that* $R'^2=a'^2+b'^2$ (§ 5·50).

The family is

$$(x-a)^2+(y-b)^2-R^2=0.$$

Calculate δR and $\{(\delta a)^2+(\delta b)^2\}^{\frac{1}{2}}$, corresponding to a positive change of δa in a, as far as terms in $(\delta a)^3$, and thus verify that neighbouring circles do not intersect.

[We find

$$\delta R=A\delta a+B(\delta a)^2+C_1(\delta a)^3+O(\delta a)^4,$$

$$\{(\delta a)^2+(\delta b)^2\}^{\frac{1}{2}}=A\,\delta a+B(\delta a)^2+C_2(\delta a)^3+O(\delta a)^4,$$

where

$$A=(a'^2+b'^2)^{\frac{1}{2}}, \quad B=(a'a''+b'b'')/2A,$$

$$C_1=\{(\Sigma a'a'''+\Sigma a''^2)A^2-(\Sigma a'a'')^2\}/6A^3,$$

$$C_2=\{(\tfrac{1}{6}\Sigma a'a'''+\tfrac{1}{8}\Sigma a''^2)A^2-\tfrac{1}{8}(\Sigma a'a'')^2\}/A^3,$$

where Σ denotes summation over the functions a and b; so that

$$\delta R-\{(\delta a)^2+(\delta b)^2\}^{\frac{1}{2}}=\frac{\Sigma a'^2\,\Sigma a''^2-(\Sigma a'a'')^2}{24A^3}(\delta a)^3+O(\delta a)^4,$$

which is positive when δa is small.]

(3) *Curvature of the envelope.* Let y', y'' ... denote the derivatives of y along the curve (a), and y_e', y_e'' ... the derivatives along the envelope. Then

$$y'=y_e'=-f_x'/f_y',$$

and the difference of the curvatures is given by

$$\frac{1}{\rho_e}-\frac{1}{\rho}=\frac{y_e''-y''}{(1+y'^2)^{\frac{3}{2}}}.$$

Differentiating $f=0$ twice along the curve (a) we obtain

$$(1) \qquad f_{xx}''+2y'f_{xy}''+y'^2f_{yy}''+y''f_y'=0;$$

and differentiating along the envelope

$$(2) \quad f_{xx}''+2y'f_{xy}''+y'^2f_{yy}''+y_e''f_y'+2a'f_{ax}''+2a'y'f_{ay}''+a'^2f_{aa}''+a''f_a'=0.$$

Subtracting (1) from (2) and using the facts that for the envelope

$$f_\alpha'=0,\quad f_{\alpha x}''+y'f_{\alpha y}''+\alpha' f_{\alpha\alpha}''=0,$$

we have

$$\frac{1}{\rho_e}-\frac{1}{\rho}=\frac{(f_x' f_{\alpha y}''-f_y' f_{\alpha x}'')^2}{f_{\alpha\alpha}''(f_x'^2+f_y'^2)^{\frac{3}{2}}}=\frac{J^2}{f_{\alpha\alpha}''(f_x'^2+f_y'^2)^{\frac{3}{2}}}.$$

We thus arrive as before at the fact that the curvatures cannot be equal unless $J=0$. Equality of curvature at a point of contact is equivalent to second order contact.

(4) Verify that, for the family of circles

$$(x-a)^2+(y-b)^2-R^2=0,$$

the condition $J=0$ reduces to $R'^2=a'^2+b'^2$.

(5) Show that the curvature of the curve whose tangential equation is $\phi(l, m)=0$ is

$$\frac{(l\phi_l'+m\phi_m')^2}{(l^2+m^2)^{\frac{3}{2}}(\phi_{ll}''\phi_m'^2-2\phi_{lm}''\phi_l'\phi_m'+\phi_{mm}''\phi_l'^2)}$$

at its point of contact with $lx+my=1$.

[The curve is the envelope of $lx+my=1$ under the condition $\phi(l, m)=0$. Take l as parameter and let m_1, m_2, ... be the derivatives of m with respect to l. We have

$$m_1=-\phi_l'/\phi_m',\quad m_2=-(\phi_{ll}''\phi_m'^2-2\phi_{lm}''\phi_l'\phi_m'+\phi_{mm}''\phi_l'^2)/\phi_m'^3.$$

The coordinates of the characteristic point satisfy

$$lx+my=1,\quad x+m_1y=0.$$

Apply the general formula of Example 3, remembering that $1/\rho=0$.]

(6) If A, B, C are functions of x and y, show that in general the envelope of $A\alpha^2+2B\alpha+C=0$ is $B^2=AC$. Examine the exceptional cases. [See § 5·20.]

(7) The envelope of the family

$$\phi(x, y, \alpha, \beta, \gamma)=0,\quad f_1(\alpha, \beta, \gamma)=0,\quad f_2(\alpha, \beta, \gamma)=0$$

is to be found in the result of eliminating α, β, γ from these equations and

$$\frac{\partial(\phi, f_1, f_2)}{\partial(\alpha, \beta, \gamma)}=0.$$

(8) The first positive pedal of a plane curve is the envelope of circles described on the radii vectores as diameters. The first negative pedal is the envelope of a straight line drawn through any point of the curve and at right angles to the radius vector to the point. [Lamb, *Infinitesimal Calculus*, 2nd ed., p. 382.]

(9) *Caustics.* The caustic of a curve C with respect to a luminous point A is defined as the envelope of the rays from A after reflection by C. Prove that the caustic is the evolute of the pedal, with respect to A, of the curve C',

similar to C, which is obtained by producing each radius vector from A to the curve C a distance equal to itself. Deduce that

$$\frac{1}{l}+\frac{1}{r}=\frac{2}{R\cos i}$$

r being the length of the incident ray AP, l the length of the reflected ray from P to the caustic, R the radius of curvature of C at P, and i the angle of incidence.

In particular, if the rays of light are parallel

$$l=\tfrac{1}{2}R\cos i,$$

and the normal to the caustic passes through the middle point of the radius of curvature of C.

[The normal to the pedal of a curve passes through the middle point of the radius vector (Ex. 8). Hence the reflected ray is normal to the pedal of C' with respect to A. Use the relation between the curvatures of a curve and its pedal (d.l.V.P., Vol. I, p. 323, Ex. 8).]

§ 5·70. **Similar problems in three dimensions.** Many three-dimensional problems on envelopes are direct extensions of the two-dimensional ones already treated. We shall consider some of the simplest.

(1) *Envelope of a family of surfaces*

$$f(x, y, z, \alpha)=0.$$

We define a *characteristic point* exactly in § 5·10 and show that it is an ordinary point of the surface satisfying

$$f=0,\quad f_\alpha'=0. \tag{5·71}$$

For certain values of α, the whole surface may be composed of characteristic points*, but in general there will be an isolated curve of characteristic points lying on each surface $f=0$. Such a curve is called **the characteristic of the surface** $f=0$. We then define the envelope of the family as *the surface generated by the characteristics.* Following the lines of §§ 5·10—5·60, we then prove that the envelope is obtained by eliminating α from $f=0$, $f_\alpha'=0$, taking care to discard the stationary surfaces, and surfaces composed of singular lines of the surfaces (α); that when neighbouring surfaces intersect, the limits of the curves of intersection are, in general, characteristics, but that envelopes may exist when neighbouring surfaces do not intersect; that a surface (α) touches the envelope at every point of its characteristic; and that a surface consisting of curves of contact with members of the family is in general the envelope (or part of it). An alternative enunciation of the last assertion is that a surface which, at every point, touches one of the surfaces of the family is, in general, the envelope (or part of it).

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* These may be called "*stationary surfaces*," if a name is required.

In particular, a surface whose tangent plane depends on one parameter, is the envelope of its tangent plane. Such a surface is, of course, a developable surface.

(2) *Envelope of a family of surfaces*

$$f(x, y, z, \alpha, \beta) = 0.$$

We define a *characteristic point* as an ordinary point of the surface whose distance from the surface

$$f(x, y, z, \alpha + \delta\alpha, \beta + \delta\beta) = 0$$

is small compared to $|\delta\alpha| + |\delta\beta|$. Such points are ordinary points satisfying

(5·72) $$f = 0, \quad f_\alpha' = 0, \quad f_\beta' = 0,$$

and are, in general, isolated, but whole surfaces or whole lines on particular surfaces may be composed of characteristic points. The surface composed of all isolated characteristic points or lines of characteristic points is called the envelope. As before, the envelope may or may not be generated by the limits of points of intersection of the surfaces

$$f(x, y, z, \alpha, \beta) = 0,$$
$$f(x, y, z, \alpha + \delta\alpha, \beta) = 0,$$
$$f(x, y, z, \alpha, \beta + \delta\beta) = 0,$$

and touches every surface of the family. Conversely, a surface touching every member of the family is, in general, the envelope (or part of it).

In particular, a surface whose tangent plane depends on two parameters (*i.e.* any surface) is the envelope of its tangent planes.

(3) *Envelope of the family of twisted curves*

(5·73) $$f(x, y, z, \alpha) = 0, \quad g(x, y, z, \alpha) = 0.$$

We define a characteristic point of the curve (α) as an ordinary point whose distance from the curve $(\alpha + \delta\alpha)$ is of order higher than the first in $\delta\alpha$. We find, by the former method, that such a point must satisfy

(5·731) $$f_\alpha' = 0, \quad g_\alpha' = 0.$$

In general, the equations 5·73 and 5·731 are incompatible (except perhaps for a number of isolated values of α), and there are no characteristic points. But if, as may happen, the equations 5·73 and

5·731 effectively reduce to three, and so can be satisfied by the continuous functions of α

$$x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha),$$

the twisted curve, of which the above functions give the parametric representation, is called the envelope of the family. We can then prove, as before, that the envelope, if it exists, touches all the curves at their characteristic points, and, conversely, that if a curve exists which at every point touches one member of the family, then this curve is, in general, the envelope (or part of it).

There is one important particular case in which the envelope in general exists, and that is the case of the characteristics of a family of surfaces depending on one parameter. The characteristics satisfy the equations

$$f = 0, \quad f_\alpha' = 0,$$

and therefore the four equations 5·73 and 5·731 reduce in this case to three

$$(5·74) \qquad f = 0, \quad f_\alpha' = 0, \quad f_{\alpha\alpha}'' = 0,$$

which in general will just determine the isolated characteristic points whose locus is the envelope. This curve which touches all the characteristics of the family, lies on the envelope and is called its *edge of regression**.

For proofs of many of these assertions and for further developments in the region of Solid Geometry the reader is referred to de la Vallée Poussin, Picard, and the paper (already quoted) by Bromwich and Hudson.

EXAMPLES V

(1) The edge of regression has, in general, second order contact with any surface of the family on the characteristic.

[Use Theorem 4·61.]

(2) Discuss the problem of the envelope of a moving sphere, whose centre lies on a twisted curve, (i) when the radius is constant (*une surface canal*), (ii) when the radius varies.

[Extend § 5·50 from circles to spheres.]

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* Fr. *l'arête de rebroussement*. Plane sections of the envelope have in general a cusp (*point de rebroussement*) where the plane cuts this curve. See Picard, vol. I, p. 322.

(3) *The wave surface.* The envelope of the plane

$$ax+\beta y+\gamma z=l,$$

where

$$a^2+\beta^2+\gamma^2=1$$

and

$$\frac{a^2}{l^2-a^2}+\frac{\beta^2}{l^2-b^2}+\frac{\gamma^2}{l^2-c^2}=0,$$

is the surface

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0, \quad (r^2=x^2+y^2+z^2).$$

(4) The direction cosines of the normal to the surface obtained by eliminating a between $f(x, y, z, a)=0$ and $g(x, y, z, a)=0$ are given by

$$l : m : n=\frac{\partial(f, g)}{\partial(x, a)} : \frac{\partial(f, g)}{\partial(y, a)} : \frac{\partial(f, g)}{\partial(z, a)}.$$

Obtain the corresponding result for a surface obtained by elimination of two parameters between three equations. Apply these results to the proof of the contact of a surface with an envelope.

(5) The envelope of

$$f(x, y, z, a, \beta, \gamma)=0,$$

where

$$\phi(a, \beta, \gamma)=0, \quad \psi(a, \beta, \gamma)=0,$$

is given by

$$f=0, \quad \phi=0, \quad \psi=0, \quad \frac{\partial(f, \phi, \psi)}{\partial(a, \beta, \gamma)}=0.$$

(6) The envelope of

$$f(x, y, z, a, \beta, \gamma)=0,$$

where

$$\phi(a, \beta, \gamma)=0,$$

is given by

$$f=0, \quad \phi=0, \quad f_a'/\phi_a'=f_\beta'/\phi_\beta'=f_\gamma'/\phi_\gamma'.$$

(7) Given a surface S, suppose that with each point m of S as centre a sphere Σ is drawn with variable radius R. Prove that this sphere Σ in general touches its envelope in two points M and M' such that MM' is normal to the tangent plane at m to S.

CHAPTER VI

SINGULAR POINTS OF PLANE CURVES*

§ 6·10. **Exceptional cases in general. Singular points.** In the course of the preceding chapters we have encountered cases of exception in which, when some particular relation is exactly satisfied, the general treatment usually applicable breaks down. These cases of exception correspond to exceptional points on the curve, usually such that there are only a finite number in any finite region, at which the curve has some peculiar property such as a stationary tangent, a point of inflexion, exceptionally high order contact with its circle of curvature, or its envelope, etc. All such points at which the curve possesses peculiar properties may be considered to be **singular points** on the curve, but it is usual to reserve this name for a particular class of exceptional points—the most important class—which consists of the points at which

$$f_x' = f_y' = 0$$

when the curve is $f(x, y) = 0$, and

$$\phi_1'(t) = \phi_2'(t) = 0$$

when the curve is $x = \phi_1(t)$, $y = \phi_2(t)$. The former case is more general, and we shall be mainly concerned with it. For the neighbourhood of such points the general existence theorem 1·51 breaks down, and further investigation is needed.

We shall again in this chapter, as in Chapters IV and V, tacitly assume the existence and continuity, in the neighbourhood of the singular point under discussion, of all the partial differential coefficients of $f(x, y)$, or differential coefficients of $\phi_1(t)$, $\phi_2(t)$, that are mentioned.

§ 6·20. **Form of $f(x, y) = 0$ in the neighbourhood of a singular point.** Without loss of generality we may suppose that the singular point is the origin, so that

$$f(0, 0) = f_x'(0, 0) = f_y'(0, 0) = 0.$$

It follows from Taylor's Theorem that in the neighbourhood of the origin $f(x, y)$ can be put in the form

$$(6·21) \qquad f(x, y) = \phi_2 + \phi_3 + \ldots + \phi_n + R_n.$$

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* This chapter follows d.l.V.P., Vol. II, Chap. IX.

In this equation

$$\phi_n = \frac{1}{n!}\left\{x^n\left(\frac{\partial^n f}{\partial x^n}\right)_0 + {}_nC_1 x^{n-1}y\left(\frac{\partial^n f}{\partial x^{n-1}\partial y}\right)_0 + \dots + y^n\left(\frac{\partial^n f}{\partial y^n}\right)_0\right\}$$
$$= a_n x^n + b_n x^{n-1}y + \dots + k_n y^n,$$

the latter form being used for the sake of shortness, and R_n is the remainder after n terms. The most useful form of R_n for our purposes is the exact form*

$$(6{\cdot}211) \qquad R_n = \frac{x^{n+1}}{n!}\int_0^1 F^{(n+1)}(tx)(1-t)^n\,dt,$$

where

$$F(x) = F(x, u) = f(x, ux) \quad (u = y/x),$$

or the corresponding form

$$(6{\cdot}212) \qquad R_n = \frac{y^{n+1}}{n!}\int_0^1 G^{(n+1)}(ty)(1-t)^n\,dt,$$

where

$$G(y) = G(y, v) = f(vy, y) \quad (v = x/y).$$

Provided that, as is generally the case, ϕ_2 does not vanish identically, the singular point is said to be *of the second order*. When $\phi_2, \phi_3 \dots \phi_{n-1}$ are identically zero, but ϕ_n is not, the singular point is said to be *of order* n.

The fundamental form of the curve will not be altered by any transformation of the axes of the type

$$x = ax' + by', \quad y = cx' + dy';$$

and the properties of the transformed function $f'(x', y')$ will be the same as those of $f(x, y)$ near $(0, 0)$. We may therefore at any stage make, without loss of generality, any such transformation that simplifies the discussion.

Certain properties of the transformed functions $F(x, u)$, $G(y, v)$ should be noted. The function $F(x, u)$ has as many orders of partial differential coefficients as $f(x, y)$ near $x = 0$ *and any finite value of* u, for all such points belong to the neighbourhood of $x = 0$, $y = 0$. A similar statement holds of $G(y, v)$ near $y = 0$, *and any finite value of* v. Any point near $x = 0$, $y = 0$, corresponds to a finite value of one at least of u or v.

If we suppose for example that u is finite, and apply 6·211, we see that in such a neighbourhood

$$(6{\cdot}22) \qquad R_n = O(x^{n+1}).$$

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* d.l.V.P., *loc. cit.*, and Vol. I, p. 432.

§ 6·30. **Nature of the curves defined by $f(x, y) = 0$.** A discussion of the following simple case will illustrate the nature of the curves defined by $f(x, y) = 0$. Let

$$f(x, y) \equiv (y - \alpha x)(y - \beta x) = 0,$$

where α and β are real constants. Then the equation $f(x, y) = 0$ defines the two straight lines

$$y = \alpha x, \quad y = \beta x.$$

These are two independent solutions of $f(x, y) = 0$, which are continuous in the neighbourhood of the singular point (0, 0). These solutions, however, are not the only functions of the type $y = \phi(x)$ which satisfy $f(x, y) = 0$.

The function (or curve)

$$y = \alpha x \ (x \textit{ rational}), \quad y = \beta x \ (x \textit{ irrational})$$

also satisfies $f(x, y) = 0$. This is a difficulty that does not occur in the non-exceptional case (Theorem 1·51). It is essentially due to the fact that in this case there are *two independent* solutions near (0, 0). We see therefore that it is necessary to make the *a priori* restriction that we are concerned only with *continuous* solutions of $f(x, y) = 0$, *i.e.* such curves as can be put in one of the forms

$$y = \phi(x), \quad x = \phi(y),$$

where $\phi(x)$ or $\phi(y)$ is a continuous function *in the neighbourhood of* $x = 0$, or $y = 0$, except perhaps *at* $x = 0$, or $y = 0$ itself. There may be an infinity of other solutions of $f(x, y) = 0$, which are of no present interest.

It is not necessary to assume *a priori* that, for example, $\phi(x)$ is continuous at $x = 0$, *i.e.* $\phi(x) \to 0$ as $x \to 0$. Since we are considering solutions of $f(x, y) = 0$ in the neighbourhood of (0, 0) it is only necessary *a priori* that zero should lie between the limits of indetermination of $\phi(x)$ as $x \to 0$. It is, however, easily proved, by establishing contradictions in the alternative case*, that as $x \to 0$, $\phi(x) \to 0$.

We therefore have to consider solutions of $f(x, y) = 0$, of the type $y = \phi(x)$ or $x = \phi(y)$, for which $\phi \to 0$ as $x \to 0$, or $y \to 0$, and ϕ is continuous near $x = 0$ or $y = 0$. Such a solution we may call *a continuous solution*, or from the geometrical point of view, a **branch** of the curve.

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* Hardy, *P. M.*, p. 192.

§ 6·310. **The fundamental property of branches through a singular point.** The necessary discussion is simplified by establishing in the first place a certain general property of any branch (*i.e.* continuous solution) of $f(x, y) = 0$, without reference to questions of existence. Questions of existence are treated later on.

THEOREM 6·311. *Any branch must touch at* $(0, 0)$ *one of the straight lines defined by*

$$\phi_n = 0,$$

where n is the order of the lowest order partial differential coefficient of $f(x, y)$ which does not vanish near $(0, 0)$.

The straight lines defined by $\phi_n = 0$ may all be put in one or other of the forms

$$y = \lambda x,$$

where λ is a root of

$$k_n \lambda^n + j_n \lambda^{n-1} + \ldots + a_n = 0, \tag{6·312}$$

or

$$x = \lambda' y,$$

where λ' is a root of

$$a_n \lambda'^n + b_n \lambda'^{n-1} + \ldots + k_n = 0. \tag{6·313}$$

If y is any branch of $f(x, y) = 0$, it is required to show that either

$$y \sim \lambda x, \tag{6·314}$$

where λ is a root of 6·312, or

$$x \sim \lambda' y, \tag{6·315}$$

where λ' is a root of 6·313. In the case of $\lambda = 0$ or $\lambda' = 0$ the corresponding relation must be replaced by $y = o(x)$ or $x = o(y)$.

The theorem may be proved by establishing contradictions in every alternative case, for if neither 6·314 nor 6·315 is satisfied, then either

(i) $$y/x \to \mu \text{ or } x/y \to \mu,$$

where μ is some finite number, zero included, which is not a root of either 6·312 or 6·313,

or (ii) there exist a constant μ, not a root of 6·312 or 6·313, and an infinite sequence of values of x, which tend to 0, such that for these values

$$y/x = \mu.$$

In either case it may be proved that* $\phi_n \sim Ax^n \; (A \neq 0)$ and at the same time* $R_n = O(x^{n+1})$, and since $\phi_n + R_n = 0$ these relations are contradictory.

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* It may of course be necessary to replace x by y in both these relations.

§ 6·40. **Singular points of the second order. Existence of branches.** We now proceed to consider questions of existence, starting with the simplest case of a singular point of the second order. This case is governed by the homogeneous quadratic form

$$\phi_2 \equiv a_2x^2 + b_2xy + c_2y^2.$$

Let

(6·41) $$\Delta = b_2^2 - 4a_2c_2.$$

If $\Delta < 0$, the straight lines represented by $\phi_2 = 0$, which provide all the possible tangents to the curve at the origin, are imaginary; if $\Delta > 0$, they are real and distinct; if $\Delta = 0$, ϕ_2 is a perfect square and $\phi_2 = 0$ represents only one straight line.

The nature of the singularity is described by the following theorem.

THEOREM 6·42. **Second order singularities.**

(1) *If* $\Delta < 0$, *the singularity is* **an isolated point.** *There is no branch (real) of the curve* $f(x, y) = 0$ *through the singularity.*

(2) *If* $\Delta > 0$, *the singularity is* **a double point with distinct tangents***. *The curve has two branches through the singularity, one touching each of the distinct lines defined by* $\phi_2 = 0$.

(3) *If* $\Delta = 0$, *the form of the curve is still uncertain but, in general, unless a particular condition is satisfied, the singularity is* **a cusp of the first species.** *The curve has two branches which tend to the singularity from one side only, and have as their common tangent there the line defined by* $\phi_2 = 0$. *The branches lie on opposite sides of the tangent.*

(1) $\Delta < 0$. This case is covered by Theorem 6·311; for since there is no real line which the branch can touch there can be no branch through the singularity.

(2) $\Delta > 0$. In this case there are two distinct real roots, and we may suppose (§ 6·20) that

$$\phi_2 \equiv xy;$$

the necessary axes may be oblique. Any branch must either touch the line $y = 0$, satisfying $y = o(x)$, or touch $x = 0$, satisfying $x = o(y)$. Consider the former case and put $y = ux$, so that $u \to 0$. The equation $f(x, y) = 0$ transforms into

$$x^2u + x^3F_1(x, u) = 0$$

or

(6·43) $$H(x, u) \equiv u + xF_1(x, u) = 0,$$

---

* Frequently called **a node**.

where the function F_1 possesses continuous partial differential coefficients near (0, 0). Moreover

$$H(0, 0) = 0, \quad \left(\frac{\partial H}{\partial u}\right)_0 = 1,$$

and therefore the Existence Theorem 1·51 may be applied. There exists therefore a unique branch

$$u = \phi_1(x)$$

such that $\phi_1(x) \rightarrow 0$ as $x \rightarrow 0$. The equation $f(x, y) = 0$ has exactly one branch

$$y = x\phi_1(x)$$

touching the line $y = 0$ at the origin. Similarly there is exactly one branch

$$x = y\phi_2(y)$$

touching the line $x = 0$ at the origin. There cannot be other branches.

(3) $\Delta = 0$. In this case ϕ_2 is a perfect square and we may suppose (§ 6·20) that

$$\phi_2 = y^2,$$

$$f(x, y) \equiv y^2 + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + R_3.$$

Any branch must touch $y = 0$ at the origin. We therefore write $y = ux$, where $u \rightarrow 0$, divide by x^2, and obtain

$$H(x, u) \equiv u^2 + x(a_3 + b_3u + c_3u^2 + d_3u^3) + x^2F_1(x, u) = 0, \tag{6·44}$$

where $F_1(x, u)$ possesses partial differential coefficients near (0, 0).

If $a_3 \neq 0$, the Existence Theorem 1·51 applies, and establishes the existence of a unique function ψ such that, near (0, 0),

$$x = u^2\psi(u) \quad (\psi(0) = \alpha \neq 0).$$

Making the substitution $x/\alpha = t^2$, it is easily seen that the origin is an ordinary double point with distinct tangents for the (t, u) curve. There exist, therefore, two branches of the (u, x) curve of the form

$$u = (x/\alpha)^{\frac{1}{2}}\, g\{(x/\alpha)^{\frac{1}{2}}\}, \quad u = -(x/\alpha)^{\frac{1}{2}}\, g\{-(x/\alpha)^{\frac{1}{2}}\};$$

and so two branches of the (y, x) curve

$$y = \pm x(x/\alpha)^{\frac{1}{2}}\, g\{\pm(x/\alpha)^{\frac{1}{2}}\}, \tag{6·45}$$

where $g(0) = 1$. It will be observed that y is real for real x when and only when $x/\alpha > 0$, *i.e.* on one side only of the origin. Further, y has opposite signs on the two branches near (0, 0), and therefore the two branches lie on opposite sides of the common tangent. Such a point is called a *cusp of the first species.*

§ 6·460. **Discussion of the doubtful case.** When $a_3 = 0$, *i.e.* $\partial^3 f/\partial x^3 = 0$ for the particular axes chosen, the above discussion does not apply, and the nature of the singularity remains doubtful. If we return to equation 6·44 giving $H(x, u)$, regard it as an equation between x and u and rearrange the terms in a new Taylor's series we obtain

$$(6{\cdot}461) \quad (u^2 + b_3 ux + a_4 x^2) + (a_5 x^3 + b_4 x^2 u + c_3 x u^2) + R_3(x, u) = 0.$$

We are concerned with branches through $(0, 0)$ and 6·461 shows that $(0, 0)$ is a singular point of the second order on the (u, x) curve. We therefore apply *de novo* the preceding discussion.

Case (1). If the quadratic form $u^2 + b_3 ux + a_4 x^2$ has no real roots, the origin is an *isolated point* on the (u, x) curve and therefore also on the (y, x) curve.

Case (2). If this quadratic form has two distinct real roots α and β, there are two distinct branches of the (u, x) curve passing through $(0, 0)$ and such that

$$u \sim \alpha x, \quad u \sim \beta x$$

respectively. There are therefore two distinct branches of the (y, x) curve through $(0, 0)$ such that

$$y \sim \alpha x^2, \quad y \sim \beta x^2.$$

It is easy to see that if one root (α say) is zero, the corresponding branch of the (y, x) curve takes the form

$$y \sim \alpha' x^r \quad (r \geqslant 3).$$

It will be observed that both of these branches are real on both sides of the origin, which may be called in this case **a double point with coincident tangents.** The two branches are distinguished by their necessarily different curvature at the origin.

Case (3). If the quadratic form is a perfect square $(u - \alpha x)^2$, we again return to the doubtful case. If, however, as is in general the case, $a_5 \neq 0$ in 6·461, the (u, x) curve, after the last section, has two branches near $(0, 0)$ touching $u - \alpha x = 0$ and such that

$$u - \alpha x \sim \pm x (\beta x)^{\frac{1}{2}} \quad (\beta \neq 0).$$

It follows that the (y, x) curve has two branches near $(0, 0)$ such that

$$y - \alpha x^2 \sim \pm x^2 (\beta x)^{\frac{1}{2}}.$$

These two branches exist only on one side of $(0, 0)$, have (of course) the common tangent $y = 0$, but now lie *on the same side of the tangent.* Such a point is called **a cusp of the second species.** Such a cusp is an essentially more complicated singularity than a cusp of the first species.

In the exceptional case, however, $a_5 = 0$, and the nature of the curve is still doubtful, for the origin is still a double point of the curve obtained by transformation from the (u, x) curve. It is necessary to start the discussion yet again at the beginning, and to continue until a decision is reached. If the curve is algebraic $f(x, y)$ is a polynomial, and the process must eventually terminate, for each step consumes more terms. If, however, $f(x, y)$ is not a polynomial, the process may never terminate, and the nature of the singularity remains undecided. In all cases in which the process terminates (whatever the stage) the resulting singularity may be classified as one or other of an isolated point, a double point (with distinct or coincident tangents), or a cusp of the first or second species.

§ 6·60. **Singular points of order n.** The discussion of such points is very similar to the case $n = 2$, and may be rapidly sketched. We have already proved that any branch touches one of the lines defined by $\phi_n = 0$. We have therefore only to consider the existence and form of the branch or branches associated with any given factor of ϕ_n. Let the given factor be y, of multiplicity $k \leqslant n$. Then $f(x, y) = 0$ takes the form

$$y^k \psi_{n-k}(x, y) + \phi_{n+1}(x, y) + R_{n+1} = 0.$$

Put $y = ux$, so that $u \to 0$ for the branches in question, and

$$u^k \psi_{n-k}(1, u) + x\phi_{n+1}(1, u) + R'_{n+1} = 0,$$

where $$\psi_{n-1}(1, 0) \neq 0, \quad R'_{n+1} = O(x^2).$$

Case (1). $k = 1$. The origin is an ordinary point of the (u, x) curve and Theorem 1·51 applies. *There exists exactly one branch touching each line defined by a simple factor of* ϕ_n.

Case (2). $k > 1$, *but* $\phi_{n+1}(1, 0) \neq 0$. Theorem 1·51 applies and shows that there exists a unique function of u such that

$$x = u^k \phi(u) \quad (\phi(0) \neq 0).$$

If k is odd arguments similar to those used in Theorem 6·42, Case (3), show that the relation between x and u can be put in the form

$$u = x^{1/k} g(x^{1/k}) \quad (g(0) \neq 0),$$

so that u is real for real x near $(0, 0)$ and changes sign with x. Hence

$$y = x^{1+1/k} g(x^{1/k}),$$

and y does not change sign with x. There is thus one branch corresponding to this factor of ϕ_n which touches it at the origin *without an inflexion.*

If k is even, we obtain in the same way

$$u = \pm (\alpha x)^{1/k} g\{\pm (\alpha x)^{1/k}\} \quad (\alpha = 1/\phi(0)).$$

Here u is real only when x has the sign of α, and y has opposite signs on the two branches. There are therefore two branches to the (y, x) curve corresponding to this factor of ϕ_n, which touch it and form a *cusp of the first species at the origin.*

Case (3). $k > 1$ *and* $\phi_{n+1}(1, 0) = 0$. In this case the (u, x) curve has a singularity of order not greater than k at the origin ($k \leqslant n$). We make a fresh start to analyse the (u, x) singularity, and proceed as before. If the curve is algebraic, the process will terminate at some stage, since $k \leqslant n$ and each step consumes more terms.

EXAMPLES VI

(1) The various species of singular points of the second order are illustrated by the following algebraic curves at the origin of coordinates.

Isolated point: $x^2 + y^2 + x^3 = 0$.

Double point with distinct tangents: $x^3 + y^3 - 3axy = 0$.

Cusp of first species: $y^2 - x^3 = 0$.

Double point with coincident tangents: $y^2(1 + x) - x^4 = 0$.

Cusp of the second species: $(y - x^2)^2 - x^5 = 0$.

(2) *Singularities of* $x = \phi_1(t)$, $y = \phi_2(t)$. A singular point is one at which $\phi_1'(t) = \phi_2'(t) = 0$. It may be assumed that the point corresponds to $t = 0$, and that $\phi_1(0) = \phi_2(0) = 0$. By suitable change of axes we can arrange that

$$x = at^p f_1(t) \qquad (p \geqslant 2, f_1(0) = 1),$$
$$y = bt^{p+\alpha} f_2(t) \quad (\alpha > 0,\ f_2(0) = 1).$$

Any existent branch touches $y = 0$.

Case (1). p *odd.* The relation between x and t can be replaced by a unique relation

$$t = \left(\frac{x}{a}\right)^{1/p} g\left\{\left(\frac{x}{a}\right)^{1/p}\right\} \quad (g(0) = 1),$$

so that t changes sign with x. We have also

$$y = b\left(\frac{x}{a}\right)^{1+\alpha/p} h\left\{\left(\frac{x}{a}\right)^{1/p}\right\} \quad (h(0) = 1).$$

If α is odd, y does not change sign with x. If α is even, y changes sign with x. In either case, there is one branch of the curve through $(0, 0)$ touching $y = 0$, with in the latter case $y = 0$ for an inflexional tangent. There is apparently no singularity, but really one of a concealed nature, for $d^2y/dx^2 \to \infty$ as $x \to 0$, if $\alpha < p$, and whatever value α has, some differential coefficient is discontinuous at $(0, 0)$.

Case (2). *p even.* We have

$$t = \pm \left(\frac{x}{a}\right)^{1/p} g\left\{\pm\left(\frac{x}{a}\right)^{1/p}\right\},$$

$$y = (\pm)^{a+p}\, b\left(\frac{x}{a}\right)^{1+a/p} h\left\{\pm\left(\frac{x}{a}\right)^{1/p}\right\}.$$

Thus t and y only exist for values of x on one side of $(0, 0)$, such that $x/a > 0$. There are two branches touching $y=0$ at $(0, 0)$ and forming a cusp of the first species if a is odd and a cusp of the second species if a is even, provided it is not the case that $h(\lambda)$ is an even function of λ. In this latter case, there is only one branch when a is even.

(3) *Radius of curvature at a cusp,*

Let P be a cusp, Q a point which tends to P along either branch of the curve, and ρ the radius of curvature at Q. Then as $Q \to P$, $\rho \to 0$, if P is the simplest type of cusp of first species.

On the other hand ρ *usually* has a finite limit (different from zero) if P is the simplest type of cusp of the second species though exceptionally $\rho \to \infty$. Consider the more complicated cases.

(4) Discuss the form of the evolute of a curve near the point corresponding to a stationary value of ρ $(d\rho/ds=0)$ on the original curve. Show that in general the evolute has a cusp of the first species.

(5) Show that in general a cusp of the second species on a given curve corresponds to a point of inflexion on the evolute.

CHAPTER VII

ASYMPTOTES OF PLANE CURVES*

§ 7·10. DEFINITION OF "$P \to \infty$." *If $P(x, y)$ is a point on the curve $f(x, y) = 0$, and if P moves along the curve so that one at least of x and y tends to $+\infty$ or to $-\infty$, then P is said to tend to infinity, and we write*

$$P \to \infty.$$

DEFINITION OF AN ASYMPTOTE. *If P be a point on the curve $y=f(x)$ and $P \to \infty$, and if the shortest distance of P from the curve*† *(or a branch of it)*

$$g(x, y) = 0$$

tends to 0 as $P \to \infty$, then the curve (or the branch of the curve) $g(x, y) = 0$ is said to be an **asymptote** *of the curve $y=f(x)$.*

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* d.l.V.P., Vol. II, pp. 391–393.

† This shortest distance will exist provided the curve (or branch) is continuous.

If the curve $g(x, y) = 0$ is a straight line

$$ax + by + c = 0,$$

then this straight line is said to be *a rectilinear asymptote of* $y = f(x)$; this name may be shortened to *asymptote* when there is no possibility of confusion.

§ 7·20. **Properties of asymptotes.** The following theorems explain the definition and are of general utility in obtaining asymptotes to a given curve.

THEOREM 7·21. *In order that* $g(x, y) = 0$ *may be an asymptote to the curve* $y = f(x)$ *it is sufficient that, as* $P \rightarrow \infty$ *along* $y = f(x)$, *the distance of* P *to* $g(x, y) = 0$, *measured parallel to a fixed direction, should tend to zero.*

For such a distance is certainly not less than the shortest distance, which therefore tends to zero.

In the particular case of a rectilinear asymptote, which is the important case, this condition is also *necessary**, if the proviso be inserted that the fixed direction is not parallel to the given straight line $g(x, y) = 0$; for then this oblique distance bears a constant finite ratio to the shortest distance.

THEOREM 7·22. *In order that the straight line* $x = a$ *may be an asymptote to the curve* $y = f(x)$ (*with coordinates rectangular or oblique*), *it is necessary and sufficient that*

$$|f(x)| \rightarrow \infty,$$

either as $x \rightarrow a + 0$, *or as* $x \rightarrow a - 0$†.

The two cases do not need separate treatment. Suppose $x \rightarrow a - 0$. If P be the point (x, y) on the curve, then $P \rightarrow \infty$ as $x \rightarrow a - 0$. Moreover, the shortest distance from P to $x - a = 0$ is $(a - x) \sin \omega$, where ω is the angle between the coordinate axes, and therefore tends to 0 as $P \rightarrow \infty$. Thus the condition is sufficient, and it is plainly also necessary.

A similar theorem could be given for asymptotes of the type $y = b$, but this case is covered by the next theorem.

THEOREM 7·23. *In order that the curve* $y = g(x)$ *should be an asymptote of the curve* $y = f(x)$, *it is sufficient* (*but not necessary*) *that*

$$f(x) - g(x) \rightarrow 0$$

as $x \rightarrow \infty$.

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* The condition is not necessary for general asymptotes. See Theorem 7·23.

† Or both, of course. $x \rightarrow a + 0$ means that $x > a$, and $x \rightarrow a$, while $x \rightarrow a - 0$ means that $x < a$, and $x \rightarrow a$.

This is merely a simplification of Theorem 7·21, so far as the sufficiency of the condition is concerned. To show that the condition is not necessary, we have only to prove that, *e.g.*, if $a > 1$, $y = x^a$ is asymptotic to $y = x^a + 1$. We have, in the first quadrant, for the former curve $x = +y^{1/a}$, and for the latter $x = (y+1)^{1/a}$. Hence the difference between the curves measured parallel to the x-axis, for a given value of y, is

$$(y+1)^{1/a} - y^{1/a},$$

which tends to 0 as $y \to \infty$, if $1/a < 1$.

In the case of rectilinear asymptotes, however, when $g(x) \equiv cx + d$, this condition is necessary as well as sufficient, as we have already stated.

Before passing on, it is well to state explicitly the following almost obvious facts, which are constantly used in the following sections.

If the straight line $y = cx + d$ is an asymptote to the curve $y = f(x)$ then

$$f(x) \sim cx, \quad f(x) - cx \to d$$

and conversely.

§ 7·240. **Asymptotes as the limits of tangents or chords.** An asymptote is sometimes defined as the limit of a tangent whose point of contact tends to infinity. Asymptotes however may exist in the sense of the definition of § 7·10 when the tangent has no limit*, as its point of contact tends to infinity. This fact shows that "the limit of the tangent" is not a suitable definition of an asymptote. The precise relations between asymptotes and tangents are defined by the following theorems.

THEOREM 7·241. *If a tangent to the curve*

$$y = f(x),$$

whose point of contact is P, has the limit

$$y = cx + d$$

as $P \to \infty$, then $y = cx + d$ is an asymptote.

The tangent at the point $P(\xi, \eta)$ is

$$y - f'(\xi)\, x - \{\eta - \xi f'(\xi)\} = 0.$$

Suppose for example that, as $\xi \to +\infty$, this tangent tends to the limiting position

$$y - cx - d = 0;$$

i.e. suppose that

$$f'(\xi) \to c, \quad f(\xi) - \xi f'(\xi) \to d.$$

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* The curve need not even have a tangent.

If $c \neq 0$, $f'(\xi) \sim c$, and therefore* $f(\xi) \sim c\xi$. Therefore

$$\frac{1}{\xi} - \frac{f'(\xi)}{f(\xi)} - \frac{d}{\xi f(\xi)} = o\left\{\frac{1}{\xi f(\xi)}\right\},$$

$$\frac{1}{\xi} - \frac{f'(\xi)}{f(\xi)} - \frac{d}{c\xi^2} = o\left(\frac{1}{\xi^2}\right).$$

On integrating we have

$$A + \log \xi - \log f(\xi) + \frac{d}{c\xi} = o\left(\frac{1}{\xi}\right),$$

or

$$\log\left[\frac{B\xi \exp(d/c\xi)}{f(\xi)}\right] = o\left(\frac{1}{\xi}\right),$$

$$\frac{B\xi \exp(d/c\xi)}{f(\xi)} = 1 + o\left(\frac{1}{\xi}\right).$$

But, as $f(\xi) \sim c\xi$, we must have $B = c$. Therefore

$$\frac{c\xi}{f(\xi)}\left\{1 + \frac{d}{c\xi} + O\left(\frac{1}{\xi^2}\right)\right\} = 1 + o\left(\frac{1}{\xi}\right),$$

and so finally

$$f(\xi) = c\xi + d + o(1);$$

or in other words

$$y = cx + d$$

is an asymptote. The case $c = 0$ alone remains to be considered.

In this case we have to prove that if

$$f'(\xi) \to 0, \quad f(\xi) - \xi f'(\xi) \to d,$$

then $f(\xi) \to d$. Let us write

$$f(\xi) = g(\xi) - \xi;$$

then

$$g'(\xi) \to 1, \quad g(\xi) - \xi g'(\xi) \to d.$$

Therefore, by what we have already proved,

$$g(\xi) = \xi + d + o(1),$$

so that

$$f(\xi) = d + o(1),$$

as was to be proved. It follows that $y = d$ is an asymptote and the proof of our theorem is completed.

If the tangent has a limiting position, it follows from the proof of the last theorem that

$$(7{\cdot}2411) \qquad f'(\xi) = c + o\left(\frac{1}{\xi}\right).$$

This condition is however not in itself sufficient to imply the existence of a limit. For example, if $f(x) = cx + \log\log x$, $f'(\xi)$ satisfies 7·2411, but no asymptote $y = cx + d$ exists.

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* By l'Hospital's theorem, d.l.V.P., Vol. I, p. 124.

The geometrical meaning of this necessary condition 7·2411 may be mentioned in passing. It is that, unless the slope of the tangent tends sufficiently rapidly to its limit, the point of intersection of the tangent and the y-axis for example will not remain within a finite distance of the origin and "the limit of the tangent will lie wholly at infinity".

THEOREM 7·242. *Under the same conditions as in the last theorem, the asymptote is the limiting position of a chord two of whose points of intersection with the curve, say P, Q, together tend to infinity in a direction in which the curve is asymptotic to its asymptote.*

Let P be (ξ_1, η_1) and Q be (ξ_2, η_2), and suppose that

$$\xi_1 \to +\infty, \quad \xi_2 \to +\infty.$$

Then the chord is

$$y - \eta_1 = \frac{\eta_2 - \eta_1}{\xi_2 - \xi_1}(x - \xi_1)$$
$$= f'(\xi)(x - \xi_1),$$

where $\xi_2 > \xi > \xi_1$, by the mean value theorem, supposing that $\xi_2 > \xi_1$. Since $f'(\xi) \to c$, we have only to prove that

$$\xi_1(f'(\xi) - c) \to 0.$$

But this follows at once from the facts that (by 7·2411)

$$\xi(f'(\xi) - c) \to 0$$

and $\xi_1 < \xi$, and so the theorem is proved.

The restrictive hypothesis needed in the two last theorems is of some interest. Its necessity may be illustrated by the curve

$$y = f(x) \equiv cx + x^{-a}\sin(x^2).$$

If $a > 0$, $y = f(x)$ has the asymptote $y = cx$ as $x \to \infty$, but $f'(x)$ has no limit as $x \to \infty$ unless $a > 1$. If $a > 1$,

$$f'(x) - c = 2x^{1-a}\cos(x^2) - ax^{-1-a}\sin(x^2) \to 0,$$

but

$$x(f'(x) - c) \to 0$$

if and only if $a > 2$. In this case only ($a > 2$) the tangent tends to the asymptote. This example illustrates what we stated above, namely that the "limit of a tangent whose point of contact P is such that $P \to \infty$" or "the limit of a chord etc." are unsuitable as definitions of a rectilinear asymptote except perhaps for algebraic curves, where the restrictions are always satisfied.

§ 7·250. The preceding definition and theorems have been stated as referring to a curve $y=f(x)$. They all apply without change to any branch of the curve $f(x, y)=0$ which satisfies the conditions of the implicit function theorem, and so can be expressed in the form

$$y=\psi(x) \quad (x>x_0), \quad \textit{or} \quad x=\psi(y) \quad (y>y_0).$$

As some such conditions are essential to enable us to assert the existence of the branch in question, they form no restriction on the generality of the foregoing discussion.

For a curve given in polar coordinates by the equation $r=f(\theta)$, the fundamental theorem is the following.

THEOREM 7·251. *If* $r\to\infty$ *as* $\theta\to\alpha$ *and if*

$$f(\theta)(\alpha-\theta)\to b,$$

then the straight line

$$r\sin(\alpha-\theta)=b$$

is an asymptote to the curve; and conversely.

The proof is left to the reader.

§ 7·30. **Asymptotes of algebraic curves.** In the case of an algebraic curve, the behaviour of any branch as $x\to\infty$ or (and) $y\to\infty$ can always be reduced, by a suitable substitution, to the study of the branches of an algebraic curve in the neighbourhood of the origin. For example, suppose both $x\to\infty$ and $y\to\infty$. The equation of the algebraic curve can be written in the form

$$(7·301) \qquad \phi_n(x, y)+\phi_{n-1}(x, y)+\ldots+\phi_0=0,$$

where $\phi_r(x, y)$ is a polynomial in x and y, homogeneous and of degree r. Write $x=1/x'$, $y=1/y'$. Then 7·301 in general transforms into

$$\frac{1}{(x'y')^n}\phi_n(y', x')+\frac{1}{(x'y')^{n-1}}\phi_{n-1}(y', x')+\ldots+\phi_0=0,$$

or

$$\phi_n(y', x')+x'y'\phi_{n-1}(y', x')+\ldots+(x'y')^n\phi_0=0.$$

We have therefore to study the form of the branches of this curve in the neighbourhood of the origin, which is a multiple point of order n. This is the problem whose solution was sketched in § 6·60. It is convenient however to develop an alternative direct method of attack, though it should always be borne in mind that the study of the asymptotes of algebraic curves is identical theoretically with the study of the form of the branches through a singular point.

When we discuss the existence of branches belonging to any possible asymptote, we shall in general reduce the problem by a substitution to the problem of the existence of a branch touching a definite straight line through the origin; for to any chosen asymptote a definite tangent at the origin is made to correspond by the substitution. This provides a convenient way of specifying precisely what is meant by a single branch or branches of the algebraic curve near infinity. The branch associated with a particular asymptote means the branch corresponding to that which touches a particular tangent at the origin of the transformed curve.

Suppose that $f(x, y)=0$ is the equation of an algebraic curve of degree n, which may be put into the form 7·301. We shall attempt to find the rectilinear asymptotes of the various branches of this curve which are not parallel to the axis of y, *i.e.* asymptotes of the form

$$y=cx+d.$$

It is clear that an exactly similar procedure, on interchanging x and y, will find for us the asymptotes of the form

$$x=cy+d,$$

i.e. those not parallel to the axis of x. The apparent exception does therefore not limit the generality of the discussion.

As $P \longrightarrow \infty$ along a branch of the curve $f(x, y)=0$ in a direction ($x \longrightarrow +\infty$ say) asymptotic to the asymptote $y=cx+d$, we have

$$y \sim cx, \quad y-cx \longrightarrow d.$$

Writing $y=tx$, $x=1/x'$ in equation 7·301 and dividing through by x^n, we have

$$(7·302) \qquad \phi_n(1, t)+x'\phi_{n-1}(1, t)+\ldots=0,$$

an equation giving t in terms of x'.

Now by hypothesis $t \longrightarrow c$ as $x \longrightarrow \infty$, *i.e.* as $x' \longrightarrow 0$, so that t remains finite as $x' \longrightarrow 0$. Moreover, it satisfies an equation of the form

$$a_0 t^\lambda+a_1 t^{\lambda-1}+\ldots+(a_\lambda+o(1))=0, \quad (\lambda \leqslant n).$$

Now the roots of such an equation are continuous functions of a_λ the constant term. Therefore, as $x' \longrightarrow 0$, $t \longrightarrow a$, where a is the root of the equation

$$a_0 t^\lambda+\ldots+a_\lambda=0,$$

i.e. of

$$(7·303) \qquad \phi_n(1, a)=0.$$

We have therefore found that *the first condition to be satisfied by the asymptote* $y=cx+d$ *is*

$$(7·31) \qquad \phi_n(1, c)=0.$$

Unless this equation has real roots there will be no asymptotes of the assumed form. Let us suppose therefore that c is a real root of this equation. Writing $y=cx+v=v+c/x'$, and substituting for y in the equation of the curve, we have

$$\phi_n(x, cx+v)+\phi_{n-1}(x, cx+v)+\ldots=0.$$

It is required that $v \longrightarrow d$, as $x \longrightarrow \infty$. On dividing through this equation by x^n, we have

$$\phi_n(1, c+x'v)+x'\phi_{n-1}(1, c+x'v)+\ldots=0,$$

and, by applying Taylor's theorem,

$$x'v\,\phi_n'(1, c)+\tfrac{1}{2}x'^2v^2\phi_n''(1, c)+\ldots+x'\,\phi_{n-1}(1, c)+x'^2\,v\,\phi'_{n-1}(1, c)+\ldots=0.$$

The number of terms in each expansion is finite, so that there is no question of validity. Suppose now that

$$\phi_n'(1, c) \neq 0,$$

i.e. that c is a simple root of the equation $\phi_n(1, t)=0$. Then v satisfies the equation

$$v\phi_n'(1, c)+\phi_{n-1}(1, c)+x'\psi(v, x')=0,$$

where ψ is a polynomial in v and x' of degree n at most. Since $v \to d$, v remains finite as $x' \to 0$, and therefore

$$\psi(v, x')=O(1).$$

Therefore, by a repetition of our preceding arguments, d must be a root of the equation

$$d\phi_n'(1, c)+\phi_{n-1}(1, c)=0,$$

i.e.

$$d=-\phi_{n-1}(1, c)/\phi_n'(1, c) \tag{7·32}$$

provided $\phi_n'(1, c) \neq 0$. We have therefore found that *if $y=cx+d$ is an asymptote then*

$$\phi_n(1, c)=0,$$

and if c is a simple real root of $\phi_n(1, c)=0$ then

$$d \doteq -\phi_{n-1}(1, c)/\phi_n'(1, c).$$

We must now consider the case

$$\phi_n'(1, c)=0,$$

in which c is a multiple root of $\phi_n(1, t)=0$. Suppose first of all that

$$\phi_{n-1}(1, c) \neq 0.$$

Then the equation satisfied by v may be written $\phi_{n-1}(1, c)+x'\psi(v, x')=0$, and as v is to remain finite as $x' \to 0$, we have

$$\phi_{n-1}(1, c)+O(x')=0,$$

which contradicts $\phi_{n-1}(1, c) \neq 0$. *There is therefore in this case no rectilinear asymptote.*

Let us now suppose that

$$\phi_{n-1}(1, c)=0.$$

Then the equation satisfied by v may be written

$$\frac{1}{2!}x'^2v^2\phi_n''(1, c)+\frac{1}{3!}x'^3v^3\phi_n'''(1, c)+\ldots+x'^2v\phi'_{n-1}(1, c)$$
$$+\frac{1}{2!}x'^3v^2\phi''_{n-1}(1, c)+\ldots+x'^2\phi_{n-2}(1, c)+\ldots+\ldots=0,$$

or

$$\tfrac{1}{2}v^2\phi_n''(1, c)+v\phi'_{n-1}(1, c)+\phi_{n-2}(1, c)+O(x')=0.$$

It follows that, provided either $\phi_n''(1, c) \neq 0$ or $\phi'_{n-1}(1, c) \neq 0$, d must satisfy the quadratic (possibly linear) equation

$$\tfrac{1}{2}d^2\phi_n''(1, c)+d\phi'_{n-1}(1, c)+\phi_{n-2}(1, c)=0; \tag{7·33}$$

that if $\phi_n''(1, c)=0$, $\phi'_{n-1}(1, c)=0$, and $\phi_{n-2}(1, c) \neq 0$, there is no rectilinear asymptote of this type; while if $\phi_n''(1, c)=\phi'_{n-1}(1, c)=\phi_{n-2}(1, c)=0$, the matter is still uncertain. We then continue the above process and, since the degree of the equation is finite, the process must eventually terminate, leaving us either no asymptotes of the assumed type, or else equations by which to determine c and d. We can sum up the results of the preceding discussion as follows.

In order that

$$y=cx+d$$

may be an asymptote of the curve

$$\phi_n(x, y)+\phi_{n-1}(x, y)+\ldots=0,$$

it is necessary that c should be a real root of

$$\phi_n(1, t)=0;$$

and that, if c is a simple root of this equation,

$$d=-\phi_{n-1}(1, c)/\phi_n'(1, c).$$

If c is a multiple root, there is no asymptote unless

$$\phi_{n-1}(1, c)=0.$$

If also $\phi_{n-1}(1, c)=0$, then d must satisfy the equation

$$\tfrac{1}{2}d^2\phi_n''(1, c)+d\phi'_{n-1}(1, c)+\phi_{n-2}(1, c)=0$$

provided this equation contains d; and so on.

A less explicit but more compact statement of these facts may be made as follows. We first of all notice that, on substituting $y=v+c/x'$ into the equation of the curve, and reducing, we eventually obtain in all cases an equation of the form

$$\psi(v, c)+x'\psi(v, c)+\ldots=0, \tag{7·331}$$

where $\psi(v, c)$ is polynomial in v and c which is not identically null, but which may or may not contain v. It follows that d must be determined by the equation

$$\psi(d, c)=0, \tag{7·332}$$

and that if this does not contain d then there is no rectilinear asymptote for this particular value of c. Then we may say that *in order that*

$$y=cx+d$$

may be an asymptote of the curve

$$\phi_n(x, y)+\phi_{n-1}(x, y)+\ldots=0,$$

it is necessary that c should be a real root of

$$\phi_n(1, t)=0,$$

and that, when a value of c has thus been determined, d should be a real root of

$$\psi(v, c)=0.$$

If there is no such real root there are no rectilinear asymptotes for this value of c.

There is yet one more necessary condition that follows from the assertion that $y=cx+d$ is an asymptote, namely that there should exist a real branch of the curve $f(x, y)=0$ of the form

$$y=cx+d+u,$$

where $u \longrightarrow 0$ as $x \longrightarrow \infty$ (or as $x \longrightarrow -\infty$ as the case may be). On substituting $d+u$ for v in equation 7·331 we have

$$(7·333) \qquad \psi(d+u, c)+x'\psi_1(d+u, c)+\ldots=0,$$

or for the sake of shortness

$$\chi(d+u, c, x')=0,$$

and the necessary condition that we require is that equation 7·333 admits at least one real root $u(x')$ such that $u(x') \longrightarrow 0$ as $x' \longrightarrow +0$ [or as $x' \longrightarrow -0$ as the case may be]. The problem is thus reduced to the ordinary problem of the existence of implicit functions, and admits the solutions of Theorem 1·51 or Chapter VI.

It may be noticed that the sign of $u(x')$ determines whether the curve lies above or below its asymptote for large values of x, and that if $u(x')$ exists and $u(x') \longrightarrow 0$ as $x' \longrightarrow +0$, or as $x' \longrightarrow -0$, the curve approaches its asymptote as $x \longrightarrow +\infty$ or as $x \longrightarrow -\infty$ respectively.

It is easily seen that, *with the addition of the last condition*, the foregoing necessary conditions for the existence of rectilinear asymptotes are also sufficient. We may therefore enunciate the following theorem, which embodies one of the usual rules for obtaining rectilinear asymptotes.

THEOREM 7·34. **Rule I for Asymptotes.** *In order that*

$$y=cx+d$$

may be an asymptote of the curve

$$\phi_n(x, y)+\phi_{n-1}(x, y)+\ldots=0,$$

it is necessary and sufficient (1) *that c should be a real root of*

$$\phi_n(1, t)=0;$$

(2) *that when c has so been chosen, d should be a real root of*

$$\psi(v, c)=0;$$

and (3) *that when d has so been chosen*

$$\chi(d+u, c, x')=0$$

should admit at least one real root $u(x')$ such that $u(x') \longrightarrow 0$ as $x' \longrightarrow +0$ or (and) as $x' \longrightarrow -0$.

Here $\psi(v, c)$ and $\chi(d+u, c, x')$ are certain polynomials which have been defined in the course of the foregoing discussion. The existence of the branch $u(x')$ may be discussed when necessary by the methods of Chapter VI.

§7·40. **Asymptotes parallel to the axes of coordinates.** It is of interest to consider in greater detail the case in which zero is a possible value of c, *i.e.* in which

$$\phi_n(1, 0)=0.$$

We note that $\phi_n(1,0)$ is the coefficient of x^n. In this case

$$d = -\phi_{n-1}(1,0)/\phi_n'(1,0) \qquad (\phi_n'(1,0) \neq 0).$$

Now $\phi_n'(1,0)$ is the coefficient of t in $\phi_n(1,t)$, and $\phi_{n-1}(1,0)$ is the term independent of t in $\phi_{n-1}(1,t)$. Hence

$$y\phi_n'(1,0) + \phi_{n-1}(1,0)$$

is the coefficient of x^{n-1}, *i.e.* of the highest power of x in $f(x,y)$. Moreover

$$y\phi_n'(1,0) + \phi_{n-1}(1,0) = 0$$

is a possible asymptote, and effectively one if condition (3), that a suitable real solution of

$$\chi(d+u, 0, x') = 0$$

exists, is satisfied. Again if $\phi_n'(1,0)=0$, and $\phi_{n-1}(1,0)=0$, d is determined by the equation (unless meaningless)

$$\tfrac{1}{2}d^2\phi_n''(1,0) + d\phi'_{n-1}(1,0) + \phi_{n-2}(1,0) = 0,$$

and the asymptotes if they exist are given by

$$\tfrac{1}{2}y^2\phi_n''(1,0) + y\phi'_{n-1}(1,0) + \phi_{n-2}(1,0) = 0,$$

where the expression on the left is the coefficient of x^{n-2}, *i.e.* of the highest power of x in $f(x,y)$. It is easy to see that this holds in general, and therefore that all possible asymptotes of the form $y=d$ may be obtained by equating to zero the coefficient of the highest power of x occurring in the equation $f(x,y)=0$. It should be observed that this rule can only be effective when the term x^n does not occur. We may therefore state the following theorem embodying this rule.

THEOREM 7·41. **Rule II for Asymptotes.** *Rectilinear asymptotes parallel to the axis of y (x) can only exist if the term in y^n (x^n) does not occur in $f(x,y)$. When they exist, they can be obtained by equating to zero the coefficient of the highest power of y (x) occurring in $f(x,y)$. Condition* (3) *must be shown to be satisfied for each line so obtained, before we may assert that such a line is actually an asymptote.*

§ 7·50. **Existence of branches of the curve, asymptotic to the asymptote.** It is important to call attention to certain general cases in which condition (3) is automatically satisfied, and in which, therefore, when we have obtained c and d we can at once assert that $y=cx+d$ is an asymptote*. This we do in the following theorem.

THEOREM 7·51. *If d is a simple root of $\psi(v,c)=0$, or in particular if c is a simple root of $\phi(1,t)=0$, then*

$$y = cx + d$$

is an asymptote, and moreover asymptotic to a single branch of the curve both as $x \to +\infty$ and as $x \to -\infty$. The branch in general lies on opposite sides of the asymptote at the two ends.

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* Cf. § 6·60.

If c is a simple root of $\phi(1, t)=0$, we have already seen that the equation determining d is linear, and therefore that d is a simple root of $\psi(v, c)=0$. When this is so, the equation determining u, namely,

$$\psi(d+u, c)+x'\psi_1(d+u, c)+\ldots=0,$$

may be written

$$\chi(u, x') \equiv u\psi'(d, c)+\tfrac{1}{2}u^2\psi''(d, c)+\ldots+x'\psi_1(d+u, c)+\ldots=0,$$

where $\psi'(d, c)\neq 0$, and the values $(0, 0)$ satisfy the equation. The origin of the curve $\chi(u, x')=0$ is therefore an ordinary point, since $\left(\frac{\partial\chi}{\partial u}\right)_0\neq 0$. It follows by the Existence Theorem 1·51 that a unique function $u(x')$ exists for values of x' such that $|x'|<k$, and that $u(x')\rightarrow 0$ as $|x'|\rightarrow 0$. Further $u(x')$ in general changes sign with x'. Condition (3) is satisfied, and our theorem is proved. This corresponds to the case $k=1$ of § 6·60.

We may apply the preceding reasoning to Rule II, and obtain the following theorem analogous to 7·51.

THEOREM 7·511. *If the highest power of x is x^{n-1}, and if its coefficient is $ay+b$ $(a\neq 0)$, then $ay+b=0$ is an asymptote, asymptotic in both the directions $y\rightarrow+\infty$, $y\rightarrow-\infty$ to a single branch of the curve.*

We may observe here that the need for condition (3) is simply in order to exclude the possibility of $u(x')$ being complex for real values of x' however small. Since $(0, 0)$ is always a point on the algebraic curve

$$\chi(d+u, c, x')=0,$$

there always exist one or more functions $u(x')$, such that $u(x')\rightarrow 0$ as $|x'|\rightarrow 0$, if complex values are admitted. In this case too, complex values of c and d may also be admitted, and to such a pair will correspond a branch of the curve (necessarily complex). The need for condition (3) arises from the fact that while to complex asymptotes correspond complex branches of the curve, to real asymptotes do not necessarily correspond real branches. The corresponding phenomenon in the case of singular points is the occurrence of *an isolated point.*

If we strike out all conditions of reality for real values of x, Theorem 7·34 takes the simpler form.

In order that $y=cx+d$ may be an asymptote of the curve

$$\phi_n(x, y)+\phi_{n-1}(x, y)+\ldots=0$$

it is necessary and sufficient (1) *that c should be a root of*

$$\phi_n(1, t)=0$$

and (2) *that when c has been so chosen d should be a root of*

$$\psi(v, c)=0.$$

In general $\phi_n(1, t)=0$ has n distinct roots. We may therefore, with due caution, say that *in general* a curve of degree n has n rectilinear asymptotes real or complex.

§ 7·60. After what precedes, the reader should have no difficulty in giving a strict proof of the validity of the following rule for the rectilinear asymptotes of an algebraic curve.

THEOREM 7·61. **Rule III for Asymptotes.** *If the equation of an algebraic curve can be expressed in the form*

$$f_n(x, y)+f_{n-2}(x, y)=0,$$

where

$$f_n(x, y)=\prod_{r=1}^{n}(a_r x+b_r y+c_r),$$

and $f_{n-2}(x, y)$ *is of degree* $n-2$ *at most in* x *and* y, *and if no factor of* $f_n(x, y)$ *is a constant, and no two factors of* $f_n(x, y)$ *represent identical or parallel straight lines, then the curve has* n *asymptotes whose equations are*

$$a_r x+b_r y+c_r=0 \qquad (r=1, 2, \dots n).$$

It should be noted that in this case all the lines are necessarily asymptotes to real branches of the curve (Theorem 7·51).

§ 7·70. **Curvilinear asymptotes.** We have seen that under certain conditions, although c is a real root of $\phi_n(1, t)=0$, yet there cannot exist any corresponding rectilinear asymptote. [In fact in such cases there cannot exist any such asymptote even if complex values are taken into consideration.] An investigation similar to the preceding shows us that in certain cases there exists a real branch of $f(x, y)=0$, which satisfies the relation $y \sim cx$, as $P \longrightarrow \infty$, but which does not satisfy the relation $y-cx \longrightarrow d$ for any finite value of d. In order to obtain a knowledge of the form of any branch of the curve $f(x, y)=0$ as $P \longrightarrow \infty$, it is necessary to undertake a further investigation of such cases. We shall find that instead of having a straight line as asymptote, the curve has, in the simplest case, a parabola as asymptote. We proceed to discuss this simplest case before passing on to the general one.

Parabolic asymptotes. The simplest case left over from the last section is that in which

$$\phi_n(1, c)=\phi_n'(1, c)=0, \quad \phi_{n-1}(1, c)\neq 0;$$

and the simplest form of this case is obtained by supposing

$$\phi_n''(1, c)\neq 0.$$

We write $y=v+cx=v+c/x'$, and note that we must have

$$(7·72) \qquad v=o(x).$$

The curve then takes the form

$$\frac{1}{2\,!}x'^2 v^2 \phi_n''(1, c)+\frac{1}{3\,!}x'^3 v^3 \phi_n'''(1, c)+\dots+x'\phi_{n-1}(1, c)+o(x')=0,$$

or

$$\tfrac{1}{2}v^2\{\phi_n''(1, c)+o(1)\}+x\{\phi_{n-1}(1, c)+o(1)\}=0.$$

In order that this equation may have a real solution it is necessary that

$$x\phi_{n-1}(1, c)/\phi_n''(1, c) < 0,$$

i.e. that $x \longrightarrow -\infty$ if $\phi_{n-1}(1, c)/\phi_n''(1, c) > 0$, and $x \longrightarrow +\infty$ if

$$\phi_{n-1}(1, c)/\phi_n''(1, c) < 0.$$

It will be sufficient to consider one of these cases, for instance the latter. We put

$$(7·73) \qquad a=\{-2\phi_{n-1}(1, c)/\phi_n''(1, c)\}^{\frac{1}{2}}.$$

Then, as $x \to +\infty$, it is necessary that either $v \sim ax^{\frac{1}{2}}$, or $v \sim -ax^{\frac{1}{2}}$, both of which hypotheses fulfil the condition 7·72. We must therefore have

$$y = cx \pm ax^{\frac{1}{2}}(1 + o(1)),$$

as $x \to +\infty$, while as $x \to -\infty$ there can be no real values of y satisfying 7·72. We have still to go a step further before we find an asymptote to the branch. Putting $v = \pm ax^{\frac{1}{2}} + u$, where $u = o(x^{\frac{1}{2}})$, we find that u must satisfy

$$\pm ax^{\frac{1}{2}} u \{\phi_n''(1, c) + o(1)\} \pm ax^{\frac{1}{2}} [\phi'_{n-1}(1, c) + \tfrac{1}{6} a^2 \phi_n'''(1, c) + o(1)] + O(1) = 0,$$

or
$$u \sim -\phi'_{n-1}(1, c)/\phi_n''(1, c) - a^2 \phi_n'''(1, c)/6\phi_n''(1, c).$$

Therefore *we must have*

$$y = cx \pm ax^{\frac{1}{2}} + \beta + o(1),$$

as $x \to \infty$, *where*
$$a = \{-2\phi_{n-1}(1, c)/\phi_n''(1, c)\}^{\frac{1}{2}},$$

and
$$\beta = -\phi'_{n-1}/\phi_n'' + \phi_{n-1}\phi_n'''/3\phi_n''^2,$$

where ϕ'_{n-1} *etc. stand for* $\phi'_{n-1}(1, c)$, *so that the two branches of the curve (if they exist) must be asymptotic to the two arms of the parabola*

$$(y - cx - \beta)^2 = a^2 x.$$

It only remains to show that if we write

$$y = cx \pm ax^{\frac{1}{2}} + \beta + w,$$

there exists in both cases a real function w for large positive values of x, such that $w \to 0$ as $x \to +\infty$; this follows at once from the fundamental theorem 1·51 in the manner of 7·50, for β is the root of a linear equation. The reader will have no difficulty in supplying the details. We have therefore proved the following theorem.

THEOREM 7·74. **Parabolic asymptotes.** *If* c *is a root of* $\phi_n(1, c) = 0$ *such that*

$$\phi_n(1, c) = \phi_n'(1, c) = 0, \quad \phi_{n-1}(1, c) \neq 0, \quad \phi_n''(1, c) \neq 0,$$

then there exist two branches of the curve

$$\phi_n(x, y) + \phi_{n-1}(x, y) + \ldots = 0$$

which possess the two arms of the parabola

$$(y - cx + \phi'_{n-1}/\phi_n'' - \phi_{n-1}\phi_n'''/3\phi_n''^2)^2 = -2x\phi_{n-1}/\phi_n''$$

for parabolic asymptotes.

The best possible parabola. Before leaving this simplest case it should be noticed that the arms of any parabola of the form

$$(y - cx - \beta)^2 = a^2 x + \mu,$$

where μ is any constant, are asymptotic to the two branches of the given curve. We can in fact determine μ in such a way as to give the closest possible approximation to the two branches, and as the parabola is in no way rendered more complicated by a value of μ other than zero, it is worth while to determine this best possible parabola. We shall find that, while in general the shortest distance between the parabola and the curve is $O(x^{-\frac{1}{2}})$, for one and only one value of μ it is $O(1/x)$.

Proceeding as in the last section we find that $w \sim \pm \gamma x^{-\frac{1}{2}}$, and that if $w = \pm \gamma x^{-\frac{1}{2}} + z$ then $z = O(1/x)$.

It follows that

$$y - cx - \beta = \pm a x^{\frac{1}{2}} \pm \gamma x^{-\frac{1}{2}} + O(1/x).$$

Moreover, if $(y - cx - \beta)^2 = a^2 x + \mu$,

$$y - cx - \beta = \pm a x^{\frac{1}{2}} \pm \tfrac{1}{2}(\mu/a) x^{-\frac{1}{2}} + O(1/x).$$

If therefore

$$\mu = 2a\gamma,$$

the ordinates of the curve and the parabola differ by $O(1/x)$, while for all other values of μ the ordinates differ by $O(x^{-\frac{1}{2}})$. It follows that *the best possible representation of these branches of the curve by a parabola is afforded by the parabola*

$$(y - cx - \beta)^2 = a^2 x + \mu,$$

where

$$\mu = 2a\gamma.$$

The general case. The following theorem covers all cases.

THEOREM 7·75. **General curvilinear asymptotes.** *If c is a root of*

$$\phi_n(1, t) = 0$$

to which no rectilinear asymptote can correspond, but to which correspond one or more real branches of the curve

$$\phi_n(x, y) + \phi_{n-1}(x, y) + \ldots = 0$$

satisfying

$$y \sim cx,$$

as $x \to +\infty$ or $x \to -\infty$ as the case may be, then any one of these branches takes the form

$$y = cx + dt^p \left\{1 + \sum_{i=1}^{p} a_i t^{-i}\right\} + O(t^{-1}),$$

where p is a positive integer, $d \neq 0$, and t is determined in one of the following ways:

(1) $$t = x^{1/q} \quad (q > p),$$

where q is odd, and the curve has one real branch of the given form both as $x \to +\infty$ and as $x \to -\infty$;

(2) $$t = x^{1/q} \quad (q > p),$$

where q is even, and the curve can have two real branches of the given form as $x \to +\infty$ and none as $x \to -\infty$;

(3) $$t = (-x)^{1/q} \quad (q > p),$$

where q is even and the curve can have two real branches of the given form as $x \to -\infty$ and none as $x \to +\infty$.

The corresponding asymptote is of course

$$y = cx + dt^p \left\{1 + \sum_{i=1}^{p} a_i t^{-i}\right\}.$$

This theorem is obviously merely a natural extension of the fundamental existence theorem, on the lines sketched in Chapter VI, especially §§ 6·40, 6·60. The details may be left to the reader.

NOTE A

A PROPERTY OF DIFFERENTIAL COEFFICIENTS

In pursuance of our general policy we define, for instance, the tangent at $P(x_0)$, to the curve $y=f(x)$, as the limit of the chord PQ when $Q \to P$. For this purpose we require merely the existence of $f'(x_0)$. The important question then arises "when does the chord Q_1Q_2 tend to the tangent at P as $Q_1 \to P$ and $Q_2 \to P$?" Similar questions occur in connection with curvature. [Theorems 2·21, 3·23.]

In general, of course, a more stringent condition than the mere existence of $f'(x_0)$ is required. But cases of *geometrical* interest occur in which actually no more stringent condition is required for two moving points than for one. In the example quoted, if $Q_1 \to P$ and $Q_2 \to P$ from opposite sides, then the chord Q_1Q_2 does tend to the tangent at P provided only $f'(x_0)$ exists.

The cases that occur all reduce to the following question: "*When can it be asserted that the existence of $f'(x_0)$ implies that*

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \to f'(x_0),$$

as $x_1, x_2 \to x_0$?" The question is answered in a simple manner by the following theorem which it is convenient to state and prove here.

If $f'(0)$ exists, and $|x_1-x_2|$ is never small compared to the smaller of x_1 and x_2, then*

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \to f'(0), \qquad \text{.................................(A)}$$

as $x_1, x_2 \to 0$. In particular if x_1 and x_2 have always opposite signs then (A) *is certainly true.*

Suppose for simplicity that $f(0)=0$. Then as $x_1 \to 0$, by definition,

$$\frac{f(x_1)}{x_1}=\frac{f(x_1)-f(0)}{x_1-0} \to f'(0),$$

since $f'(0)$ exists. Therefore $f(x_1)=x_1 f'(0)+o(x_1)$. Similarly,

$$f(x_2)=x_2 f'(0)+o(x_2),$$

and therefore
$$\frac{f(x_1)-f(x_2)}{x_1-x_2}=f'(0)+o\{x/|x_1-x_2|\},$$

where x is the larger of $|x_1|$ and $|x_2|$. If now, as $x_1, x_2 \to 0$, $|x_1-x_2|>kx$ for some fixed positive value of k, then $o\{x/|x_1-x_2|\}=o(1/k) \to 0$, and therefore

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \to f'(0).$$

---

* *I.e.* there exists a constant k, independent of x_1 and x_2, such that

$$|x_1-x_2|>k|x_1|, \quad |x_1-x_2|>k|x_2|.$$

If in particular x_1 and x_2 have opposite signs, which is the geometrically interesting case, then $|x_1-x_2|>|x_1|$, $|x_1-x_2|>|x_2|$; the extra condition is automatically satisfied, and the mere existence of $f'(0)$ is sufficient to ensure that

$$\frac{f(x_1)-f(x_2)}{x_1-x_2}\rightarrow f'(0).$$

NOTE B

THE REMAINDER IN TAYLOR'S THEOREM

We have made use in Note A of the equation $f(x)=xf'(0)+o(x)$, or more generally

$$f(x)=f(0)+xf'(0)+o(x),$$

as equivalent to the existence of $f'(0)$. Similar equations are used frequently in this tract when we wish to avoid unnecessary assumptions, for such equations just contain all the information provided by the hypotheses. In general we can obtain an O- or o-result for the remainder in Taylor's expansion with less assumptions than are required for the use of any of the standard forms of remainder.

For example, suppose that $f^{(n)}(0)$ exists. This is equivalent to the equation

$$f^{(n-1)}(x)=f^{(n-1)}(0)+xf^{(n)}(0)+o(x).$$

Integrating this equation from 0 to x, we obtain, by L'Hospital's theorem,

$$f^{(n-2)}(x)=f^{(n-2)}(0)+xf^{(n-1)}(0)+\tfrac{1}{2}x^2f^{(n)}(0)+o(x^2).$$

If we repeat the integration $n-2$ times more, we get

$$f(x)=\sum_0^n\frac{1}{r!}x^rf^{(r)}(0)+o(x^n);$$

in other words we have proved that

$$R_{n+1}=o(x^n).$$

For this o-result, $R_{n+1}=o(x^n)$, the only hypothesis required is the existence of $f^{(n)}(0)$. In the same way we can prove that *for the O-result, $R_{n+1}=O(x^{n+1})$, the only hypothesis required is that all the upper and lower derivates of $f^{(n)}(x)$ should be bounded at $x=0$.*

To obtain a result at least as good as either the o-result or the O-result from, for example, Lagrange's remainder form, we must assume that $f^{(n+1)}(x)$ exists over an interval containing $x=0$.

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L. C. YOUNG

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PREFACE

THE object of small treatises of this kind is to enable the general student to gain rapid access to the various branches of Modern Mathematics, thereby preventing this science from breaking up into a number of disconnected parts, each belonging to its own specialist and closed to the outsider. Mathematics must form a whole, any progress in one of its parts stimulating advance in the others and raising new problems; when a branch is severed from the tree, it dies.

In writing this book, I have therefore tried above all to simplify the work of the student. On the one hand, practically no knowledge is assumed (merely what concerns existence of real numbers and their symbolism); on the other hand, the ideas of Cauchy, Riemann, Darboux, Weierstrass, familiar to the reader who is acquainted with the elementary theory, are used as much as possible.

I have also hopes that it will be of some use to the initiated, who may find here new points of view and greater generality in the treatment, owing to the idea of integration with respect to a function in space of n-dimensions. I have not however included what Hobson and others call the Fundamental Theorem of the Integral Calculus, namely the connection with the Theory of Derivation.

The Theory of Integration, which forms the subject of this book, has long been one of the most useful tools of Mathematics. Its methods were already employed with success by the ancient Greeks, in their investigations about Areas and Volumes. They possessed the method of exhaustion, the method of series. They were very clear about the idea of limit and this perhaps made them suspicious of the unsound method of infinitesimals, as results thus obtained were always established independently.

After the Dark Ages the rediscovery of this last method and the use of the symbolism of Algebra rendered possible the creation of the Calculus by Newton and Leibnitz. Unfortunately, *believing they had reduced everything to symbols*, they did not realise the need of examining the *ideas* these represented and testing their soundness. They conceived their Calculus to be purely formal; limiting process, a mere operation on symbols. They were very clear as to the properties they expected of such operations: possibility of operation, existence of inverse operation, reversibility of order of two consecutive operations. But they were not so clear as to the properties implied of the entities operated on, which in their case were *functions*, loosely defined as numbers depending on variable quantities.

This conception of Mathematics persisted for a long time. In the nineteenth century, however, the feeling that Mathematics is not the entire property of the mere Calculator, but rather that of the Thinker, revived at last. The result was the development of Geometry, the Theory of Groups, the Theory of Vectors, the systematic use of the Imaginary.

The Mathematicians of that century naturally also perceived the need of reforming the Infinitesimal Calculus. The reform was started with Cauchy's Theory of Limits, based on Inequalities. Cauchy also introduced the notion of *Continuity* and attempted to use it as a foundation for the Calculus. He saw the unsatisfactoriness of the notion, hitherto adopted, of Integration as Inverse Differentiation: a definition which is not constructive requires a theorem ensuring the existence and unicity of the entity in question. Cauchy defined Integration for a continuous function by an always possible limiting process and he proved that it could be considered as the inverse of Differentiation.

But the occurrence of discontinuous functions in certain simple problems and the discovery, by Weierstrass, of continuous non-differentiable functions,—by Riemann, of discontinuous integrable functions, showed that continuity is both inconvenient and unnatural for the foundation of the Calculus.

The Theory of Integration and that of Differentiation have since been built up separately as parts of the New Calculus, the *Calculus of Real Functions*, whose great generality, far from being due to a love of complication on the part of its founders, as was at one time asserted, is to be attributed to the simplicity and straightforwardness of its methods

This New Calculus would never have been possible but for the wonderful ideas of Cantor, at first completely unintelligible to the Mathematicians of the Older School, some of whom even wilfully misunderstood them and sought to lead others into error with regard to them; but which, fortunately, very much influenced a few younger men since become famous. Only too often have ideas of the greatest value been left for a long time unheeded, while their authors remain in obscurity. Galois, the founder of the Theory of Groups, was ploughed at the entrance examination of the École Polytechnique through knowing more than his examiners, and this was only the first of a series of disappointments which embittered his short life. Grassmann, the creator of the theory of vectors, remained a schoolmaster most of his life, and his book, the "Ausdehnungslehre," was burned by the publishers, who could find no buyers.

Cantor's Theory of Sets of Points and Borel's improvement of the theory of content or measure of such sets paved the way to the semi-geometric definitions of Integration, given almost simultaneously by Lebesgue and Young. These definitions represent an extension comparable to that of Arithmetic on the introduction of irrational numbers. They are substantially equivalent to the more direct one here adopted, later given by Young, using the work of Baire on functions.

My father had long thought of writing a connected account of his theory. In carrying out this task myself at his suggestion, I have tried to do justice to his ideas and to introduce a few minor improvements of my own. If I have succeeded in my endeavours, it will have been largely owing to his encouragement, and to the constant assistance of my mother and of my sister Miss R. C. H. Young.

L. C. Y.

September 1926.

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CHAPTER I

THE METHOD OF MONOTONE SEQUENCES

§ 1. Successions of numbers*.

A set of numbers is said to be *bounded* if there exist a finite number greater and a finite number less than all those of the set. The smallest number such that no number greater than it belongs to the set is called the *upper bound* of the set of numbers; the greatest number such that no smaller number belongs to the set is called the *lower bound* of the set. If there is no finite number greater than all those of the set, the set of numbers is said to be *unbounded above*, and we shall agree to say that its upper bound is $+\infty$; similarly, if there is no finite number less than all those of the set, the set of numbers is said to be *unbounded below*, and its lower bound will be $-\infty$.

A countably infinite set of numbers, written down in a definite order, is called a *succession*, e.g.

$$a_1, a_2, \ldots, a_n, \ldots.$$

Repetition of a number is allowed.

A succession is said to be *monotone ascending* if each term is greater than or equal to the preceding,

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant \ldots;$$

a succession is said to be *monotone descending* if each term is less than or equal to the preceding,

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots.$$

In either case the succession is said to be monotone, or to be a *monotone sequence*. One of the bounds of a monotone sequence is clearly its first term a_1. The other is called the *unique limit* of the monotone sequence, whether finite or infinite.

Given any succession S

$$a_1, a_2, \ldots, a_n, \ldots,$$

the succession S_r obtained from it by leaving out its r first terms, has its bounds K_r, k_r lying between those of the succession S_{r-1}, i.e. we have

$$k_{r-1} \leqslant k_r \leqslant K_r \leqslant K_{r-1}.$$

Thus, the succession of the upper bounds

$$K_1, K_2, \ldots, K_n, \ldots$$

---

* We are dealing with real numbers; $+\infty$ and $-\infty$ are regarded as distinct.

is monotone descending. Its unique limit is called the *upper limit* of the succession S. Similarly, the unique limit of the monotone ascending sequence of the lower bounds is called the *lower limit* of S.

The upper and lower limits of a monotone sequence are equal and coincide with its unique limit.

For one succession of bounds coincides with the given sequence and the other consists of terms all equal to the unique limit.

A succession whose upper and lower limits are equal is called a *sequence*, and their common value is called the *unique limit* of the sequence. A sequence having a finite limit is said to converge; otherwise it diverges. A succession which is not a sequence is said to oscillate.

Given any succession S, we call subsuccession of S a succession of numbers all belonging to S and occurring in the same order as in S. We call subsequence of S any subsuccession of S which is a sequence. The unique limit of any subsequence of S is called *a limit* of S.

THEOREM. The upper and lower limits of a succession are limits of that succession.

It is sufficient to prove this for the upper limit.

If the numbers K_n all belong to the succession, the theorem is obvious.

If any one of them does not belong to the succession, all the following are equal to it, and it is the upper limit*. Let r_0 be its index, r_1 that of the first term after a_{r_0} which is greater than a_{r_0}, r_2 that of the first term after a_{r_1} which is greater than a_{r_1}, and so on. Then the subsuccession of S,

$$a_{r_0},\ a_{r_1},\ \dots,$$

is a monotone sequence whose unique limit is K_{r_0}, because, by construction, no term of S after a_{r_0} can exceed this limit which, being the upper bound of a subsequence of S_{r_0}, cannot exceed K_{r_0}.

THEOREM. The upper and lower limits of a subsuccession of S lie between those of S.

Let S' be the given subsuccession. Let S_r' be the succession obtained from S' by leaving out its r first terms, and let S_r be obtained similarly from S.

Then S_r' is a subsuccession of S_r, and its bounds K_r' and k_r' lie between those of S_r, K_r, and k_r,

$$k_r \leqslant k_r' \leqslant K_r' \leqslant K_r.$$

This holds for all r; the theorem follows.

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[*] Omitting from a set of numbers a finite number of its elements none of which coincide with the upper bound, does not affect the upper bound.

Corollary. All the limits of a succession lie between its upper and its lower limits*.

As a kind of converse of the preceding theorem, we have:

Theorem. If a finite number of subsuccessions of S together contain all the elements of S, they have among their upper and lower limits those of S.

It is sufficient to prove this for the case of two subsuccessions and to consider only the upper limits. Let U be the upper limit of S. Then there is a subsequence having U as limit and we can so arrange that it belongs entirely to one of our subsuccessions (see footnote*). U is therefore a limit of that subsuccession. By the preceding theorem and its corollary it must therefore be its upper limit.

e-definitions of limits and convergence. To define by this method the upper limit of a bounded succession S,

$$a_1, a_2, \dots, a_n, \dots,$$

U is said to be the upper limit if given any positive number e however small, an index N can be found such that from and after $n = N$,

$$a_n \leqslant U + e,$$

while an infinite succession of n's can be found such that

$$a_n \geqslant U - e.$$

Similar definition for lower limit.

Thence the e-definition of limit of a convergent sequence

$$U - e \leqslant a_n \leqslant U + e,$$

for all n from and after N.

This method, which was known to the Greeks, is probably familiar to the student, who will easily prove the equivalence of the definitions so obtained with our former ones.

The characteristic advantage of our method is *to reduce the consideration of all successions to that of monotone sequences.*

§ 2. Successions of functions.

Corresponding to any set of numbers, we have on the straight line a set of points†; we need only choose an origin, a sense, and a unit of length.

We shall say a point is a *limiting point* of our set of points, and that the corresponding number is a limit of the set of numbers, if every interval of which it is the centre contains an infinite number of points of the set. This agrees with our definition of limit in the case of a succession.

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* In particular, all the limits of a sequence coincide with its unique limit.

† This is equivalent to what is called the Cantor-Dedekind axiom.

A set containing all its limiting points is said to be *closed.* A point belonging to a set and not a limiting point is called an *isolated point* of that set.

After the finite sets, consisting of a finite number of points, and the sequences and successions of points, the simplest sets are the *intervals.* An interval consists of all the points between its endpoints. If it includes these it is closed, if neither, *open.*

Corresponding to any pair of numbers, we have a point in the plane; corresponding to any set of pairs of numbers, a set of points in the plane. Similarly any set of numbers given n by n, may be taken to represent a set of points *in n dimensions.* We may agree to represent the n coordinates of a point by *a single symbol,* and let x stand for the ensemble of the n numbers x_1, x_2, …, x_n.

Two points $a=(a_1, a_2, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ such that each coordinate of a is less than the corresponding one of b, define an interval (a, b), consisting of all the points x whose coordinates all lie between the corresponding coordinates of a and b, that is the set of points belonging to the rectangle or to the n-dimensional parallelopiped whose sides are parallel to the axes and which has a and b for its endpoints. The interval is open if it consists only of the internal points, closed, if of all the points. We shall call length of an interval the length of a principal diagonal. It is convenient to define a *stretch* to be either an *open* interval or the limiting case of an open interval when one or more of the inequalities between the coordinates of a and b are replaced by equalities. Thus in one dimension a stretch would be either an open interval or a point; in two dimensions it would be an open interval, or an open side of an interval, or a point, etc.

When distinction is necessary, we use the symbol t to denote a point on the straight line.

Definition of a function. A quantity y is said to be a function of x, in an interval (a, b), if to each x of that interval corresponds a single value y.

We use the symbol $f(x)$ to denote a function of x.

A function is said to be *bounded* if the set of numbers consisting of all its values is bounded.

If corresponding to each point x we are given a succession of numbers $f_1(x)$, $f_2(x)$, …, these may be said to define a *succession of functions.*

The upper bound $K(x)$ of the numbers corresponding to the point x define a new function, which we call the *upper bounding function* of the succession of functions.

Similarly we define the *lower bounding function.*

The upper limit $U(x)$ of the numbers $f_1(x)$, $f_2(x)$, … corresponding to the point x defines a new function, which we call the *upper function* of the succession of functions.

Similarly we define the *lower function.*

The succession of functions is called a *sequence*, if its upper and lower functions are identical. They are then called the *limiting function* of the sequence.

The succession of functions is said to be a *monotone ascending sequence* if the numbers corresponding to each x form an ascending sequence; it is said to be *monotone descending* if they form a monotone descending sequence. In either case the succession of functions is called a *monotone sequence of functions.*

Given any succession of functions, the upper bounding function $K_r(x)$ of the succession obtained by leaving out the r first functions, generates as r increases a monotone descending sequence which converges to the upper function $U(x)$, and the corresponding monotone ascending sequence of the lower bounding functions, converges to the lower function.

We have thus again reduced the consideration of all successions of functions to that of monotone sequences of functions.

In the case of monotone sequences of functions one bounding function is the first function, the other is the limiting function.

The convergence of a sequence of functions is said to be *bounded* if the bounding functions are bounded. The functions of the sequence are then said to be uniformly bounded.

The convergence of a sequence of functions $f_n(x)$ to a limiting function $f(x)$ in a closed interval is said to be *uniform* if, given any positive number e, an index N independent of x can be found such that, for all n from and after N,

$$|f_n(x) - f(x)| < e \text{ for all } x.$$

It is uniform in an open interval if uniform in every closed component interval. (See Appendix, 1, p. 44, last 18 lines and seq.)

§ 3. **Limits of a function at a point.** The upper and lower bounds of a set of numbers consisting of all the values of a function in an interval are called *the upper and lower bounds of the function in that interval.* Between them lie the bounds of the function in any interval interior to that interval.

Let x_0 be any point interior to the interval of definition (a, b). Let $w_1, w_2, \ldots, w_r, \ldots$ be any succession of intervals having x_0 as internal point and whose lengths converge to zero, each interval being contained in the preceding. Let M_r, m_r denote respectively the upper and lower bounds of the values of the function in w_r, *excluding the point* x_0. As r increases they form two monotone sequences of numbers and their limits, U and L respectively, are called the *upper* and the *lower limits* *of the function at the point* x_0. These limits are independent of the choice of the succession of intervals[*].

For consider two successions of intervals $w_1, w_2, \dots$ and $w_1', w_2', \dots$; let the corresponding limits be U, L and U', L'.

There cannot be more than a finite number of intervals of the second succession not interior to w_r. Since r is arbitrary U', L' therefore lie between U and L.

But, reversing the rôles of the two successions, U, L lie between U', L'. Hence $U = U'$, $L = L'$.

If in the definition of upper and lower limits at a point x_0 instead of intervals containing the point x_0, we consider intervals (open or closed) having x_0 for a corner-point and as before each contained in the preceding and with length decreasing to zero—this defines corresponding to each quadrant[†] (open or closed) at x_0 a unique pair of *upper* and *lower limits* (or limits of approach) *in that* (open or closed) *quadrant.*

In the case of an internal point of the interval of definition the upper and lower limits at the point are respectively equal to the greatest and to the least of the upper and lower limits in the various *closed* quadrants at the point. We may use this to define the upper and lower limits at every point of the *closed* interval of definition.

We thus have, in all cases, corresponding to each point x of the interval of definition, three numbers, the value of the function and its upper and lower limits.

It is obvious that the greatest and the least[‡] of these three numbers are respectively the limits of the upper and lower bounds of the function in any succession of intervals of lengths decreasing to zero, the point x being internal to all of them (the value at x is not excluded). If the

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[*] Also each interval need not lie in the preceding provided they ultimately shrink up to x_0. For let u_1 be the smallest interval containing all those of the succession, u_2 the smallest interval containing all except w_1, etc. Then $u_1, u_2, \dots$ are each inside the preceding and shrink up to x_0. Again if v_1 is w_1, v_2 the largest interval inside w_2 which has no points outside $w_1, \dots, v_n$ the largest interval in w_n having no points outside v_{n-1}, then $v_1, v_2, \dots$ are a succession of intervals each inside the preceding and shrinking up to x_0. Also clearly

$$u_n \geqslant w_n \geqslant v_n.$$

Hence the limits for w_n lie between the corresponding limits for u_n and v_n which coincide.

[†] By quadrant at a point we mean, in one dimension, one or other of the two sides of the point; in n dimensions, an angle determined by n parallels to the axes through the point.

[‡] They are sometimes called the maximum and minimum limits and their difference is sometimes called the *jump* of the function at the point x.

value at the point is the greatest, it is also the limit of the upper bounds of the function in any succession of closed intervals which have x as corner-point, and whose lengths decrease to zero. In that case the function is said to be *upper semicontinuous* at the point x. Similarly, if the value of the function is the least of the three numbers, the function is said to be *lower semicontinuous* at the point.

Similarly we may define upper and lower semicontinuity in a closed or open quadrant at a point.

If a function is both upper and lower semicontinuous at a point and its value there is finite, it is said to be *continuous** at the point. Otherwise the function is *discontinuous* at the point.

§ **4. Semicontinuity and the theorem of bounds.** A function is said to be upper semicontinuous in an interval if it is upper semicontinuous at every point of the interval. We shall call it a *U-function.*

A function is said to be lower semicontinuous in an interval if it is lower semicontinuous at every point of the interval. We shall call it an *L-function.*

In either case it is said to be *semicontinuous* in the interval. A function which is both an L and a U and assumes only finite values is said to be *continuous.*

e-definition of semicontinuity at a point. A function $f(x)$ is said to be upper semicontinuous at the point x_0 if, given any positive quantity e, there is an interval having x_0 as middle point throughout which if $f(x_0)$ is finite

while, if $f(x_0)$ is $-\infty$,

$$\left.\begin{aligned} &f(x) \leqslant f(x_0) + e, \\ &f(x) \leqslant -1/e \end{aligned}\right\} (U).$$

A function $f(x)$ is said to be lower semicontinuous at the point x_0 if, given any positive quantity e, there is an interval having x_0 as middle point, throughout which, if $f(x_0)$ is finite,

while, if $f(x_0)$ is $+\infty$,

$$\left.\begin{aligned} &f(x) \geqslant f(x_0) - e, \\ &f(x) \geqslant 1/e \end{aligned}\right\} (L).$$

If $f(x_0)$ is $+\infty$, then $f(x)$ is certainly upper semicontinuous at x_0; if $f(x_0)$ is $-\infty$, then $f(x)$ is certainly lower semicontinuous at x_0.

Theorem. An L-function assumes its lower bound in every closed interval; a U-function, its upper bound.

Divide the given interval into two equal parts. If m is the lower bound of our L-function in the given closed interval, it is also its lower bound in at least one of these closed subintervals. Let W_1 be the first having this property. Again divide W_1 into two equal parts, and let W_2 be the first of these closed subintervals of W_1 in which m is again the lower bound. And so on.

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* See Appendix, 1.

The succession of intervals

$$W_1, W_2, \ldots, W_n, \ldots$$

consists of closed intervals, each contained in the preceding, and whose lengths decrease to zero. There is exactly one point common to all of them. The value of our L-function at that point is therefore the limit of its lower bounds in the succession of intervals, that is to say m. Q.E.D.

Similarly, we may prove the corresponding theorem for U-functions.

The theorem of bounds.

If $f_1(x), f_2(x), \ldots$, is a monotone ascending sequence of functions having $f(x)$ as limiting function; if u_n and l_n are respectively the upper and the lower bounds of $f_n(x)$ in a fixed closed interval, u and l those of $f(x)$ in the same interval, then

$$\lim u_n = u \quad \text{while} \quad \lim l_n \leqslant l.$$

Moreover, if the functions are all L-functions, then

$$\lim l_n = l.$$

It is obvious that if f_1 is never greater than f_2, the same is true of their bounds. Hence

$$u_1 \leqslant u_2 \leqslant \ldots \leqslant u; \quad l_1 \leqslant l_2 \leqslant \ldots \leqslant l.$$

Therefore $$\lim u_n \leqslant u; \quad \lim l_n \leqslant l.$$

But if A is any quantity less than u, there are points x such that $f(x) > A$. At any such point x, we have, since $f_n(x)$ converges to $f(x)$, $f_n(x) > A$ from and after a certain $f_N(x)$. Therefore, a fortiori,

$$\lim u_n > A,$$

or, since A was any quantity less than u, $\lim u_n \geqslant u$.

Therefore $$\lim u_n = u, \quad \text{while} \quad \lim l_n \leqslant l.$$

In the case in which the functions f_n are L-functions, we can find a point x_n where f_n assumes its lower bound. Let x' be any limiting point of the x_n, and let B be any quantity less than $f(x')$. Then

$$f_n(x') > B$$

from and after a certain $f_{N'}$. There is therefore an interval surrounding x' throughout which, since $f_{N'}$ is an L-function, $f_{N'}(x)$ is greater than B. In that same interval, by monotony,

$$f_n(x) > B$$

from and after $f_{N'}$. In this interval there are an infinite number of the points x_n with indices greater than N'. Therefore, if N'' is the first of these, then

$$f_{N''}(x_{N''}) = l_{N''} > B.$$

Therefore, a fortiori,

$$\lim l_n \geqslant B.$$

Since B was any number less than $f(x')$,

$$f(x') \leqslant \lim l_n,$$

and, a fortiori,

$$l \leqslant \lim l_n.$$

CHAPTER II

THE GENERATION OF FUNCTIONS

§ **1. The simple functions.** The simplest functions are the constants; the value of y is the same for all x.

The next simplest functions are *the functions constant in stretches*, whose interval of definition is the sum of a finite number of stretches inside each of which the function is a finite constant. By a stretch we mean, as explained on p. 4, an open interval or a kind of limiting case of an open interval, such as a point.

A function constant in stretches is not in general semicontinuous.

For example, the function defined in the interval (0, 1), whose value in the open interval (0, 1/2) is 0, whose value at the point 1/2 is 1/2, whose value in the open interval (1/2, 1) is 1, is not semicontinuous at the point 1/2.

A function constant in stretches will certainly be lower semicontinuous at every point of its interval of definition if its value in every stretch which is not an interval is equal to the least of the values in the neighbouring intervals. It is then called a *simple L-function.*

Similarly, we call *simple U-function* a function constant in stretches whose value in every stretch which is not an interval is the greatest of the values in the neighbouring intervals.

These two types of *simple functions* have the following properties:

(i) The sum of two functions of the same type is of that type.

(ii) The function equal to the greater of two functions of the same type, and the function equal to the smaller of two functions of the same type, at each point, are of that type.

(iii) Change of sign transfers each type to the twin type.

These properties we shall refer to as *the three fundamental properties of class.*

§ 2. The generation of general semicontinuous functions by monotone sequences of simple functions.

THEOREM. Any L-function bounded below is expressible as the limit of a monotone ascending sequence of simple L-functions and also as the limit of such a sequence of simple U-functions.

Let $f(x)$ be the given L-function, defined in the interval (a, b). Let m be its lower bound.

We divide (a, b) into two equal parts and we call m'_1, m''_1 the lower bounds of $f(x)$ in each of these parts, the common boundary points being taken to belong to both.

We again bisect each of the parts.

(In the case of several variables we bisect in turn the range of each of them.)

Let m'_2, m''_2, m'''_2, m''''_2, be the bounds so obtained, and so on.

If at any stage, the nth say, one of these numbers be infinite, we replace it by the greatest of all the preceding plus n. In that case $f(x)$ would have to be infinite $(+\infty)$ in the whole of the corresponding interval.

Let $a_0(x)$, $b_0(x)$ be the constant m.

Let $a_1(x)$, $b_1(x)$ be respectively the simple L-function and the simple U-function which are equal to m'_1 in the first half of (a, b) and to m''_1 in the second half, their values at the common boundary points* being of course respectively the smallest and the largest of these two numbers.

Let $a_2(x)$, $b_2(x)$ be respectively the simple L-function and the simple U-function equal to m'_2 in the first quarter of (a, b), to m''_2 in the second, and so on.

The two sequences of functions,

$$a_0(x),\quad a_1(x),\quad a_2(x), \dots,$$
$$b_0(x),\quad b_1(x),\quad b_2(x), \dots,$$

are monotone ascending. The values of their limiting function at any point x are in both cases equal to the limits of the lower bounds of $f(x)$ in one or more successions of closed intervals each contained in the preceding and whose lengths tend to zero, and such that x is common to all of them.

Since $f(x)$ is semicontinuous the limiting functions therefore both coincide with $f(x)$. Q.E.D.

Similarly we can establish the corresponding

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* At which alone they differ.

THEOREM. Any U-function bounded above is expressible as the limit of a monotone descending sequence of simple U-functions, and also as the limit of such a sequence of simple L-functions.

The converse of these theorems is not true. Monotone sequences of simple functions do not in general define semicontinuous functions. It is, however, easy to show that monotone ascending sequences of simple L-functions always define L-functions, while monotone descending sequences of simple U-functions always define U-functions. More generally, we have

THEOREM. The limiting function of a monotone ascending sequence of L-functions is an L-function.

Let $f_n(x)$ be the generic function of the sequence, and $f(x)$ the limiting function. Since f is never less than f_n, the same is true of their lower bounds in any interval and consequently of their lower limits at any point. Let l be the lower limit of $f(x_0)$ at the point x_0. Since f_n is semicontinuous, $f_n(x_0)$ is smaller than or equal to the lower limit of f_n at x_0. Therefore, a fortiori,

$$f_n(x_0) \leqslant l.$$

This is true for all n. Therefore

$$f(x_0) \leqslant l.$$

Similarly we can prove the corresponding

THEOREM. The limit of a monotone descending sequence of U-functions is itself a U-function.

The two types of semicontinuous functions are easily seen to possess the *three fundamental properties of class*.

i. The sum of two functions of the same type is of that type.

If f and g are two L-functions bounded below they are the limits of two monotone sequences of simple L-functions

$$f_1, f_2, \dots, \quad g_1, g_2, \dots,$$

their sum is therefore the limit of the monotone ascending sequence

$$f_1+g_1, \quad f_2+g_2, \dots$$

of simple L-functions, i.e. an L-function*.

If f and g are not bounded below, every point, where neither assumes the value $-\infty$, is internal to an interval where both are bounded below.

At a point where one assumes this value, the other being different from $+\infty$, the sum assumes the value $-\infty$, and is consequently lower semicontinuous.

At a point where one function assumes the value $-\infty$, and the other the value $+\infty$, the sum is not defined.

Similarly we prove the theorem for U-functions.

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* The same reasoning establishes the property for any type of functions defined by monotone sequences of functions belonging to a type which has the property.

ii. The function which is the greater of two functions of the same type is of that type; so is the function which is the lesser of two functions of the same type.

Let f and g be two L-functions bounded below, h the function which is equal to the greater (or to the smaller) of the two at each point. f and g are the limits of monotone ascending sequences of simple L-functions

$$f_1, f_2, \ldots, f_n, \ldots,$$

$$g_1, g_2, \ldots, g_n, \ldots.$$

At each point, h is the limit of the monotone ascending sequence of simple L-functions

$$h_1, h_2, \ldots, h_n, \ldots,$$

each of which is equal to the greater (or the smaller) of the two corresponding functions at each point. h is therefore an L-function.

Now let f and g be unbounded below.

Every point where neither is equal to $-\infty$ is internal to an interval in which the above holds.

A point where f is not $-\infty$, is internal to an interval where f is greater than a finite number m. The greater of f and g is not affected if we replace by m all the values of g in that open interval which are less than m; neither does this affect the lower semicontinuity of g.

In the remaining cases h is obviously lower semicontinuous.

Similarly the theorem is proved for U-functions.

iii. Change of sign transfers a type to the twin type.

If f is the limit of a monotone ascending sequence of simple L-functions f_n then $-f$ is the limit of the monotone sequence of simple U-functions $-f_n$.

At a point where f is $-\infty$, $-f$ is $+\infty$.

These properties are also easy to establish independently. The method adopted has the advantage of being applicable to a number of other theorems.

We have seen that all bounded semicontinuous functions are expressible as the limits of monotone sequences of simple functions.

We shall next show that all functions it has been found possible to define up to this date are expressible as the limits of a *system* of monotone sequences of these functions.

§ 3. The generation of new functions. Monotone sequences of semicontinuous functions fall into four classes. Two of these have been already seen to define semicontinuous functions, namely ascending sequences of L- and descending sequences of U-functions.

The remaining two classes usually define functions which are not semicontinuous.

DEFINITION. The limit of a monotone descending sequence of L-functions is called an *UL-function*. That of a monotone ascending sequence of U-functions is called a *LU-function*.

The prefix U to the name of a function denotes the limit of a *descending* sequence of functions of that name. The prefix L to the name of a function denotes the limit of an *ascending* sequence of functions of that name. According to this nomenclature,

$$LL = L, \quad UU = U.$$

THEOREM. An L-function is both a LU and an UL. So is a U-function.

We know already that an L is a LU (p. 10). It is also an UL, because a sequence of functions each equal to the following is a particular case of a monotone descending sequence.

There are functions which are both LU and UL without being either L or U, e.g. the function constant in stretches of p. 9. There are also functions which are LU's without being UL's.

The two types of functions possess the *three fundamental properties of class.* (See footnote, p. 11.)

Monotone sequences of functions of one of these types again fall into four classes. Their limiting functions are, according to our nomenclature, LLU, ULU, LUL, UUL. It is easy to show that only two of these are new types of functions.

Let $f_1 \leqslant f_2 \leqslant \dots \to f$ be any monotone ascending sequence of LU-functions, and f their limiting function.

f_1 is the limit of an ascending sequence

$$f_{11} \leqslant f_{12} \leqslant \dots \to f_1$$

of U-functions. f_2 is the limit of an ascending sequence

$$f'_{21} \leqslant f'_{22} \leqslant \dots \to f_2$$

of U-functions. Since $f_2 \geqslant f_1$, it is also the limit of the monotone ascending sequence

$$f_{21} \leqslant f_{22} \leqslant f_{23} \leqslant \dots \to f_2$$

of U functions such that $f_{2,r}$ is the greater of $f_{1,r}$ and $f'_{2,r}$ at every point. Similarly f_3 is the limit of a monotone ascending sequence

$$f_{31} \leqslant f_{32} \leqslant \dots \to f_3$$

of U-functions, such that $f_{3,r} \geqslant f_{2,r}$.

Proceeding in this way, we get

$$\begin{array}{ll} f_{11} \leqslant f_{12} \leqslant \dots \to & f_1 \\ f_{21} \leqslant f_{22} \leqslant \dots \to & f_2 \\ \dots\dots\dots\dots\dots & \downarrow \\ \dots\dots\dots\dots\dots & f \end{array}$$

The monotone ascending sequence

$$f_{11} \leqslant f_{22} \leqslant \dots$$

has all its terms, and therefore its limit also, less than or equal to f. On the other hand, each of its terms from the rth onwards is greater than or equal to the corresponding term of the sequence which has for limit f_r. It follows that

$$f_{11} \leqslant f_{22} \leqslant \dots \to f,$$

and since the functions are all U-functions, this proves that f is a LU-function, i.e. $LLU = LU$. And similarly $UUL = UL$.

Theorem. A LU-function is both an ULU and a LUL. So is an UL.

A LU is obviously an ULU. It is a LUL being an L-U, i.e. an L-UL, since a U is an UL.

The *three fundamental properties of class* hold for the two types.

It is clear that we can go on defining functions in this way *ad infinitum*. At each stage we have four kinds of monotone sequences, and only two of these define new types of functions. Each type of function is a particular case of both types of the next stage. At each stage the three fundamental properties of class hold for the two types.

For the sake of brevity, we call U_n and L_n the two types of the nth stage, n being the number of sequences necessary to generate either, starting with simple functions.

Two new types of functions may then be obtained as follows. Let $f_1, f_2, \dots$ be a monotone sequence such that f_1 belongs to one of the types of the first stage, f_2 to one of the second stage, and so on. Its limiting function will in general belong to none of the previous types. The two types thus defined we call U_ω and L_ω; monotone sequences of these give again two more general types. And so on.

§ **4. Functions defined by other methods.** For defining a function in a given interval, the only means at our disposal are the operations of arithmetic and that of passage to the limit applied to known functions, such as the variables x and constants. This may be further complicated by choosing different laws in different parts of the fundamental interval. As we are using the operation of passage to the limit, we may always suppose these parts to consist of a finite number of stretches.

The operations of arithmetic and of passage to the limit are called *analytic operations*. A function is said to be *representable analytically*, or to be capable of an *analytic expression*, if its value at each point of its interval of definition may be obtained by means of the same analytic operations on the variables x and on constants. When considering analytic operations, we can leave out those of subtraction and division, as the former may be replaced by multiplication by a constant (-1) and addition, the latter by a power series.

THEOREM. Analytic operations performed on functions defined by a system of monotone sequences of simple functions lead to functions of the same kind.

(*a*) The operation of passage to the limit merely involves expressing the function as the limit of a sequence of functions already defined. Let these be

$$f_1, \quad f_2, \dots.$$

The monotone ascending sequence

$$g_1, \quad g_2, \dots,$$

where $g_1 = f_1$, and g_n is the greater of f_n and g_{n-1}, by property ii consists of functions of the same type as the former sequence. Its limit is the upper bounding function of that sequence, and belongs at most to the next stage. The same is true of the upper bounding functions of the sequences obtained by leaving out the r first functions f. These upper bounding functions form a monotone sequence whose limit is the required function. This proves our theorem for the operation of passage to the limit.

(*b*) If f is a function obtainable by monotone sequences, the same is true of its product by a function constant in stretches. Its product by any other function obtainable by monotone sequences may then be obtained by operations of passage to the limit.

(*c*) The operation of addition has already been seen not to affect a type. Q.E.D.

Since the variables x and the constants already belong to our set of functions, the same is therefore true of functions representable analytically. The same is true of functions having different analytical expressions in different stretches and of the limits of systems of sequences of such functions.

Other functions, theoretically existent, cannot be defined by the means at our disposal and do not occur in analysis.

CHAPTER III

MONOTONE FUNCTIONS

§ 1. **Monotone functions of one variable.** A function of one variable is said to be *monotone increasing* in its interval of definition if, as t increases, it never decreases, *monotone decreasing* if it never increases.

In either case it is a *monotone function* of the variable t. A constant is obviously a monotone function, and the only monotone function of both types.

Like functions constant in stretches, monotone functions are not in general semicontinuous. A monotone ascending function is however easily seen to be upper semicontinuous on the left and lower semicontinuous on the right at every point.

A monotone function of one variable has *unique limits on either side* at each point.

It is sufficient to prove this for monotone increasing functions, and for the left-hand side.

Let t_0 be any point of the interval, and $t_1, t_2, \dots, t_n, \dots$ a monotone ascending sequence of numbers converging to t_0. And let $f(t)$ be the given function.

The sequence of values

$$f(t_1),\ f(t_2),\ \dots,\ f(t_n),\ \dots$$

is monotone ascending. We denote its limit by $f(t_0-0)$. This limit is independent of the choice of the sequence.

For suppose another sequence give a different limit; with the points of both sequences we could form a mixed sequence, still monotone ascending, which would not give a unique limit.

Now $f(t_1)$ is the lower bound of $f(t)$ in the open interval (t_1, t_0), $f(t_2)$ in (t_2, t_0), and so on. $f(t_0-0)$ is thus the limit of the lower bounds on the left, that is, the lower limit on the left. And as $f(t_0-0)$ is greater than all the values of f at points preceding t_0, it is not less than the upper limit on the left. This proves our statement.

§ 2. **Monotone functions of two variables.** Given a function of two variables $g(x, y)$ we call *simple* increment the difference

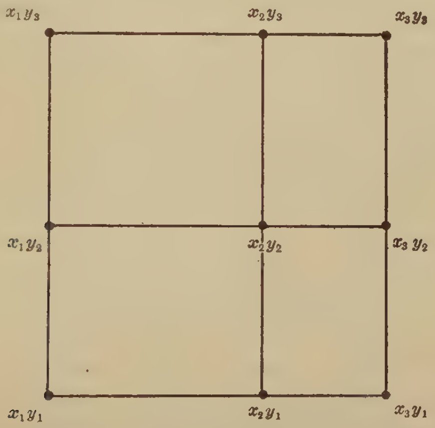
$$g(x+h, y+k)-g(x, y),$$

when neither h nor k is negative; we call *double increment* the expression[*]

$$\begin{bmatrix} g(x+h,\ y+k)-g(x+h,\ y) \\ \quad -g(x,\ y+k)+g(x,\ y) \end{bmatrix}$$

when h and k are positive. The double increment is said to be taken over the interval whose endpoints are (x, y), $(x+h,\ y+k)$. It is easy to verify that the double increment possesses the following *additive property*:

If we divide a given interval into subintervals, the sum of the double increments of $g(x, y)$ over these subintervals is the double increment over the total interval.



$$\left\{\begin{array}{l} \quad g(x_2y_2)-g(x_2y_1)-g(x_1y_2)+g(x_1y_1) \\ +g(x_3y_2)-g(x_3y_1)-g(x_2y_2)+g(x_2y_1) \\ +g(x_2y_3)-g(x_2y_2)-g(x_1y_3)+g(x_1y_2) \\ +g(x_3y_3)-g(x_3y_2)-g(x_3y_2)+g(x_2y_2) \end{array}\right.$$
$$=g(x_3y_3)-g(x_3y_1)-g(x_1y_3)+g(x_1y_1).$$

A function of two variables is said to be *monotone increasing* in its interval of definition, if its simple *and its double*[†] increments are never negative. It is said to be *monotone decreasing* if these increments are never positive.

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[*] Or, "increment with respect to y of the increment with respect to x."

[†] When only the simple increments are never negative, the function is sometimes called monotone increasing with respect to each variable separately. The functions defined in the text are sometimes called "entirely" monotone. If each of the three increments $g(x+h, y)-g(x, y)$, $g(x, y+k)-g(x, y)$, and the double increment have a constant sign, but this sign is not the same for the three increments, then the function is "pseudo-monotone."

For these monotone functions of two variables we have the following:

THEOREM*. A monotone function has a unique limit of approach in each open quadrant at every point.

We remark that an interval is determined when we are given any two opposite corner-points (or "diagonal" points) not necessarily endpoints. Let us denote by

$$[PQ]$$

the interval thus determined by P and Q, and by

$$\Delta[P, Q]$$

the double increment corresponding to the endpoints of $[P, Q]$. Let us suppose our function monotone increasing. It follows from our definition of such a function and from the additive property of the double increment that if Q lie in $[O, P]$, then

$$\Delta[O, Q] \leqslant \Delta[O, P].$$

When P describes a sequence of points

$$P_1, P_2, \ldots, P_n, \ldots \rightarrow O,$$

such that P_n always lies inside $[O, P_{n-1}]$, and tends to O, the double increments

$$\Delta[O, P_n]$$

form a monotone sequence and therefore have a unique limit.

This limit is independent of the sequence of points, provided they belong to the same open quadrant as the preceding sequence. For suppose another sequence give a different limit; with the points of both sequences we could form a mixed sequence with the same property as above (P_n lying inside $[O, P_{n-1}]$), which would not give a unique limit. The double increment thus has a unique limit of approach in each open quadrant. Since it consists of four terms, one of which remains constant and two of which have unique limits by the one-dimensional theorem, its fourth term also has a unique limit.

It follows immediately that this is the unique limit of approach of $g(x, y)$ in that open quadrant at O. Q.E.D.

§ 3. **Monotone functions of n variables.** Similarly to our definitions in the case of two variables, we may define simple, double, ..., n'ple increments of a function of the ensemble of n variables, corresponding to non-negative increments of these variables.

A function of the ensemble of n variables will be called *monotone increasing* if all these increments are non-negative, and *monotone decreasing* if none of them are positive.

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* This is equally true (by the same argument) of a pseudo-monotone function.

We then have the theorem, proved in exactly the same manner:

A monotone function has a unique limit of approach in each open quadrant at every point*.

§ 4. Total increment of a monotone function over a stretch. Given a monotone function of n variables, and an open interval inside its interval of definition, the n'ple increments over closed intervals contained in it and tending to it, have a unique limit which is also their upper bound. This limit is called the *total increment* over the open interval.

A stretch which is not an interval is the limit of open intervals containing it. We call *total increment* over such a stretch the limit of the total increments corresponding to such intervals. This limit is also their lower bound.

In order to make the matter clear, let us consider the two-dimensional case.

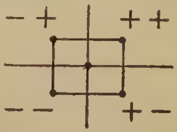
The total increment over the open interval whose endpoints are x, y and $x+h, y+k$ will be the expression

$$\left\{\begin{array}{ll} g(x+h-0;\ y+k-0) & -\,g(x+h-0;\ y+0) \\ -\,g(x+0;\ y+k-0) & +\,g(x+0;\ y+0) \end{array}\right.$$

when $g(x+0;\ y+0)$, $g(x+0;\ y-0)$, $g(x-0;\ y+0)$, $g(x-0;\ y-0)$ denote respectively the unique limits of $g(x, y)$ in the open $(+, +)$, $(+, -)$, $(-, +)$, $(-, -)$ quadrants at x, y.

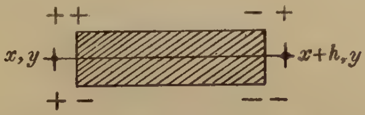
The only other kinds of stretches are points and open sides of intervals. The total increment over a point x, y will be the expression

$$g(x+0;\ y+0)-g(x-0;\ y+0)$$
$$-g(x+0;\ y-0)+g(x-0;\ y-0).$$



The total increment over the stretch x, y to $x+h, y$ will be

$$g(x+h-0;\ y+0)-g(x+h-0;\ y-0)$$
$$-g(x+0;\ y+0)+g(x+0;\ y-0),$$



and the total increment over the stretch x, y to $x, y+k$

$$g(x+0;\ y+k-0)-g(x+0;\ y+0)$$
$$-g(x-0;\ y+k-0)+g(x-0;\ y+0).$$

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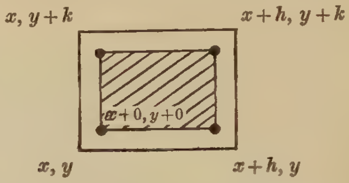
* As before, it holds equally for pseudo-monotone functions.

It is convenient to think of the symbols $x-0$, $y-0$ etc. *as representing actual points* and our stretches as ordinary closed intervals with the corresponding endpoints. The total increment over a stretch is then identical in form with the double increment.



The additive property is clearly independent of the actual geometrical existence of the points of division.

It follows that *if we divide up a stretch into stretches, the total increment over the given stretch is equal to the sum of the corresponding total increments.*

§ 5. Integration of functions constant in stretches with respect to a monotone increasing function.

Given any stretch, we define the *integral over that stretch of the constant unity* as the total increment of the given monotone increasing function over that stretch.

The integral over that stretch of a function which has there a constant value C is to be the product of this value into the total increment.

The integral over an open interval (a, b) *of a function constant in stretches* there defined is to be the sum of its integrals over the stretches of the open interval (a, b) in which it is constant.

We use the symbol

$$\int_a^b f(x)\, dg(x)$$

to denote the integral over the open interval (a, b) of the function $f(x)$ with respect to the monotone increasing function $g(x)$. The function to be integrated is called the *integrand*, the function with respect to which we integrate the *integrator*.

The integral over a stretch is defined just in the same way as the integral over an open interval, and is always expressible as such an integral possibly in a space of less dimensions.

The integrals of functions constant in stretches have the following obvious properties, which we shall call the fundamental properties of integration.

(I) If C is any constant,

$$\int_a^b C \,.\, f(x) \,.\, dg(x) = C \int_a^b f(x)\, dg(x).$$

(II) The integral of the sum of two functions is equal to the sum of their integrals

$$\int_a^b (f_1+f_2)\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$$

(III) If f_1 is greater than or equal to f_2 throughout (a, b), then

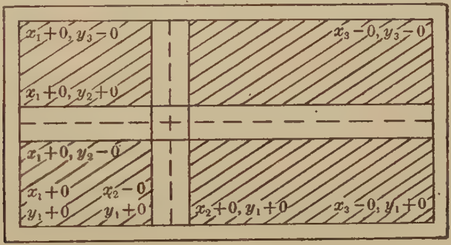
$$\int_a^b f_1\,dg \geqslant \int_a^b f_2\,dg.$$

(IV) If an interval be divided up in any manner into stretches the integral over that interval is the sum of the integrals over the partial stretches. This is really contained in (II): take $f_1=f$ in a certain stretch and zero outside, $f_2=0$ where $f_1=f$, and $f_2=f$ where $f_1=0$.

If we divide up our interval at a point we have in one dimension

$$\int_a^b f\,dg = \int_a^\xi f\,dg + \int_\xi^b f\,dg + f(\xi)\,\{g(\xi+0)-g(\xi-0)\},$$

while in two dimensions we have



$$\begin{aligned}\int_{x_1y_1}^{x_3y_3} f\,dg = {} & \int_{x_1y_1}^{x_2y_2} + \int_{x_1y_2}^{x_2y_3} + \int_{x_2y_1}^{x_3y_2} + \int_{x_2y_2}^{x_3y_3} \\ & + \int_{x_1}^{x_2} f(x, y_2)\,d\,[g(x, y_2+0)-g(x, y_2-0)] \\ & + \int_{x_2}^{x_3} f(x, y_2)\,d\,[g(x, y_2+0)-g(x, y_2-0)] \\ & + \int_{y_1}^{y_2} f(x_2, y)\,d\,[g(x_2+0, y)-g(x_2-0, y)] \\ & + \int_{y_2}^{y_3} f(x_2, y)\,d\,[g(x_2+0, y)-g(x_2-0, y)] \\ & + f(x_2, y_2)\cdot\left[\begin{array}{l} g(x_2+0, y_2+0)-g(x_2+0, y_2-0) \\ -g(x_2-0, y_2+0)+g(x_2-0, y_2-0)\end{array}\right],\end{aligned}$$

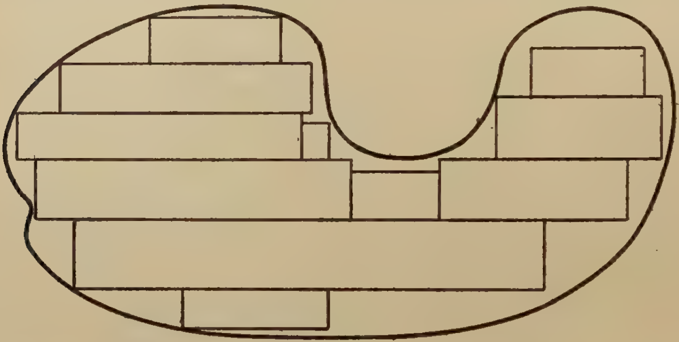
i.e. the sum of four two-dimensional integrals, four one-dimensional integrals and one, so to speak, 0-dimensional integral.

CHAPTER IV

THE INTEGRATION OF FUNCTIONS

§ 1. **Methods of the theory of integration.** The theory of integration arose from the problem of finding the area of plane curves. This led at once to the consideration of functions constant in stretches.

The original method of evaluating areas, a method known to the Greeks, consisted in attempting to construct a system of abutting rectangles such that the boundary of the area covered by them should approximate to the given curve. The problem then reduced to that of the area of rectangles.

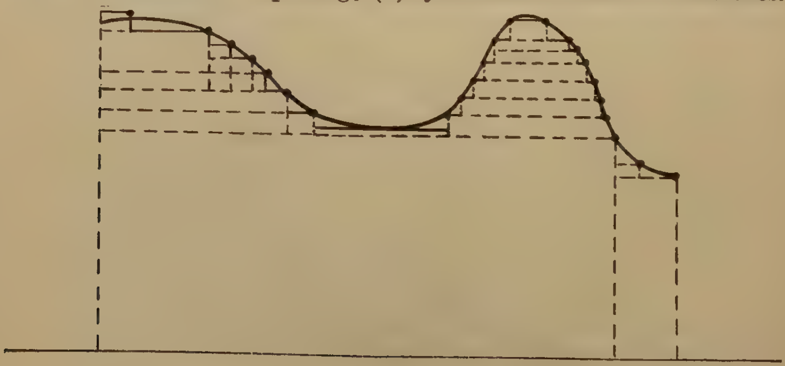


In the case of a closed curve consisting of the graph

$$y = f(x)$$

in (a, b), of the rectilinear segments $x = a$ and $x = b$, and the axis of x, it is clearly simplest to take the rectangles to have sides parallel to the axes.

This is equivalent to replacing $f(x)$ by a function constant in stretches.



The method we shall proceed to develop is a natural extension of the above. We have defined integration with respect to a monotone increasing function for functions constant in stretches; in the simplest case, in one dimension and when the integrator is the variable itself, the integral of a function constant in stretches $f(t)$,

$$\int_a^b f(t)\,.\,dt,$$

is the area of the closed curve obtained as stated above. We are therefore led to define, by successive approximations, the integral of a function which is the limit of a function constant in stretches; functions for which this definition proves valid will form the next class of integrable functions and can be used in their turn to approximate more general functions and to define their integrals; and so on. We have only to prove that at each stage the integrals of the functions by which we approximate tend to a unique limit. Such is indeed the case.

Historically the functions whose integrals with respect to a monotone increasing function were first defined were continuous. To these we easily extend integration.

Theorem. If an ascending sequence of simple L-functions and a descending sequence of simple U-functions have the same limiting function, the limit of their integrals is the same.

We suppose all the functions defined in the closed interval (a, b). The theorem is then an immediate consequence of the theorem of bounds.

The differences between corresponding functions form a monotone descending sequence of simple U-functions

$$c_1(x), \quad c_2(x), \ldots$$

having the limit zero. By the theorem of bounds their upper bounds u_n also have the limit zero. It follows from property (III) of p. 21 that

$$\int_a^b c_n(x)\,.\,dg(x) \leqslant U_n\,.\int_a^b dg(x) \to 0.$$

The theorem then follows by property (II) of the same page.

Definition. Given any continuous function we can always construct a monotone ascending sequence of simple L-functions and a monotone descending sequence of simple U-functions of which it is the limit. The limit of the integrals is, by the above theorem, independent of the choice of these sequences. (Given two pairs of sequences, we need only compare the U-sequence of one pair with the L-sequence of the other.) This limit is defined to be the integral of the given continuous function.

We have thus defined the integral of a continuous function with respect to a monotone increasing function. Such an integral is called a *Stieltjes integral.*

Stieltjes, however, only considered the case of one variable, whereas the method of monotone sequences is independent of the number of variables concerned. When the integrator is a product of these variables*, our definition of integral will be found to reduce to that of *multiple integral* such as employed for very special types of functions by *Cauchy* and *Riemann*.

§ 2. **Darboux's theorem.** A modification of the method of finding areas is the following: we construct a system of abutting rectangles all internal to the given closed curve; the upper bound of the areas corresponding to all such systems is then a lower evaluation for the required area; similarly we construct a system of abutting rectangles containing the given curve; the lower bound of such areas is then an upper evaluation.

These considerations led Riemann to frame his definition of integration. Let $f(x)$ be any function defined in (a, b) and bounded in that interval. It follows from property (III) of p. 21 that the upper bound of the integrals of the functions constant in stretches which are nowhere greater than $f(x)$ is less than or equal to the lower bound of those which are nowhere less than $f(x)$. If that property is to be maintained when we extend the definition of integration to $f(x)$, the integral of $f(x)$ will have to lie between these bounds. We have thus got a lower evaluation and an upper evaluation for the integral of $f(x)$. When these are equal they give a definition of the integral of $f(x)$.

We can also get (less good) evaluations by considering a particular set of functions constant in stretches. It is not unnatural to restrict ourselves to simple functions. For the purpose of calculation the following method is convenient: we divide up the interval (a, b) in any manner into a finite number of stretches; the simple L-function, constant in these stretches, whose value in each open interval of constancy is the lower bound of $f(x)$ in the corresponding closed interval, is clearly nowhere greater than $f(x)$; let s be its integral. The upper bound of all such numbers s we shall call *Darboux's lower evaluation* of the integral of $f(x)$ with respect to $g(x)$. Similarly the simple U-function obtained by doing the same with the upper bounds of $f(x)$ is nowhere less than $f(x)$; let S be its integral. The lower bound of all such numbers S we shall call *Darboux's upper evaluation* of the integral.

When these two bounds are equal they give *Riemann's definition of the integral of* $f(x)$; this definition is however quite insufficient for our purposes; in fact, unless the integrator is continuous, Darboux's evaluations do not even coincide in the case of simple functions.

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* The total increment of the integrator over any stretch is then the area, or volume or etc. of that stretch.

Given a subdivision of the interval (a, b) we shall say of another subdivision that it is consecutive to it, if it is got by dividing up the stretches of the first subdivision.

The simple L-function corresponding to the consecutive subdivision is obviously nowhere less than the former one. Corresponding to a system of consecutive subdivisions we thus have a monotone ascending sequence of simple L-functions. Darboux was able to prove that, in the case of a continuous integrator, the *integrals of this sequence tend to his lower evaluation*, provided the length of the longest stretch of the subdivision tends to zero. This theorem is equally true when the integrator is not assumed to be continuous*.

For our purpose it is sufficient to prove Darboux's theorem in the case where $f(x)$ is an L-function.

Given any function constant in stretches, a simple L-function and a simple U-function are uniquely determined coinciding with it in those stretches of constancy which are open intervals. We shall call them the *related* simple L- and U-functions.

If $f(x)$ is an L-function bounded below (as we are only concerned with the lower evaluation, f need not be bounded above) the simple L-functions of Darboux's theorem tend to f, and so do also their related simple U-functions (p. 10). We have only to prove the following lemma.

LEMMA. If two monotone ascending sequences of simple L-functions and also the related simple U-functions all have the same limiting function, then the integrals of the simple L-functions of the two sequences have the same limit.

Let $a_n(x)$, $b_n(x)$ be the generic terms of the two sequences of simple L-functions, $a_n'(x)$, $b_n'(x)$ the related simple U-functions.

Let $f(x)$ be the limiting function.

Suppose all these functions defined in the closed interval (a, b).

Let $b_N(x)$ be any one of the $b_n(x)$; let d be any one of the open intervals in which it is constant and D the interval got by closing d.

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* A similar proposition holds for the upper evaluation.

As a matter of fact both follow from the theorems of this chapter. For the limit of the given ascending sequence of simple L-functions is at any point the minimum limit at the point. Hence it is sufficient to prove, as we do further on, that if two monotone ascending sequences of simple L-functions have the same limiting function, then the limits of their integrals coincide.

In D, $f(x)$ is not less than b_N' and its lower bound not less than the lower bound of b_N', which coincides with the upper bound of b_N, being the common constant value in d.

By the theorem of bounds applied to the first sequence, if m denote any number smaller than the lower bound of $f(x)$ in D, we can find an n_0 from and after which, in D, the lower bound of $a_n(x)$ is greater than or equal to m. Since we may choose for m the upper bound of b_N in D, minus e, we have, a fortiori, throughout D,

$$a_n(x) \geqslant b_N(x) - e$$

for all n greater than n_0. Corresponding to each D, we have a similar inequality.

If n_1 denote the greatest of the corresponding n_0's, we have, throughout (a, b),

$$a_n(x) \geqslant b_N(x) - e$$

for all n greater than n_1.

By properties (I) and (II), remembering that e is arbitrary, we deduce, passing to the limit

$$\lim_{n=\infty} \int_a^b a_n(x)\, dg(x) \geqslant \lim_{n=\infty} \int_a^b b_n(x)\, dg(x).$$

But, reversing the rôles of the sequences, we get similarly,

$$\lim_{n=\infty} \int_a^b a_n(x)\, dg(x) \leqslant \lim_{n=\infty} \int_a^b b_n(x)\, dg(x).$$

Hence

$$\lim_{n=\infty} \int_a^b a_n(x)\, dg(x) = \lim_{n=\infty} \int_a^b b_n(x)\, dg(x). \qquad \text{Q.E.D.}$$

COROLLARY I. The above lemma also holds for integrals over a stretch.

(For an integral over a stretch which is not an open interval is expressible as an integral over an open interval of a space of less dimensions.)

COROLLARY II. If the limiting function is a simple L-function the limit of the integrals is the integral of the limiting function.

Case 1. f is a constant. We take for one of our sequences the repetition of the function itself.

Case 2. f is any simple function. We divide up the interval into the stretches where f is constant. In each of these, by case 1, the limit of the integrals is equal to the integral of the limiting function. By property (IV), p. 21, the proposition follows by addition.

THEOREM. (Darboux's theorem for an L-function.) Darboux's lower evaluation of the integral of an L-function bounded below is the unique limit of the numbers s corresponding to any system of consecutive subdivisions of (a, b) when the length of the longest stretch of the subdivision tends to zero.

Let f_n be the simple L-function corresponding to the nth subdivision of (a, b). By the method employed to prove the theorem of p. 10, f_n, and also the related simple U-function f_n', tend to the given L-function f. Now consider any two systems of consecutive subdivisions. The corresponding two sequences of simple L-functions fulfil the conditions of our lemma, and therefore the two monotone ascending sequences of numbers s have the same limit, which is their common upper bound. This limit is therefore unique and is the upper bound of all the numbers s.

§ 3. **The integration of semicontinuous functions.** The preceding theorem enables us to define integration for any L-function bounded below, in accordance with the methods of § 1. Since Darboux's lower evaluation is the limit of the integrals of certain monotone sequences of simple L-functions which tend to $f(x)$, it gives a natural definition of the integral of $f(x)$.

DEFINITION. The integral of an L-function bounded below is Darboux's lower evaluation.

We shall say a sequence of functions is integrable term by term if the limit of the integrals (with respect to a given monotone increasing function) equals the integral of the limiting function.

THEOREM. Every monotone ascending sequence of L-functions, bounded below, is integrable term by term.

Let $f_1, f_2, \dots, f_n, \dots$ be the sequence and f the limiting function. The integral of f_1 is defined by means of the auxiliary monotone ascending sequence of simple L-functions

$$f_{11}, f_{12}, f_{13}, \dots, f_{1n}, \dots$$

which tend to f_1, as do also the related simple U-functions.

Similarly f_2 is the limit of an auxiliary ascending sequence of simple L-functions

$$f_{21}, f_{22}, f_{23}, \dots, f_{2n}, \dots$$

and of the related simple U-functions. Since f_1 is nowhere greater than f_2 it is clear that the latter is also the limit of the simple L-function which is at every point equal to the greater of f_{1n} and f_{2n}, and also of the simple U-function which is the greater of their related simple U-functions. We can consequently always so arrange that

$$f_{2n} \geqslant f_{1n} \text{ for all } n.$$

Proceeding in this way, we get, as on p. 26, a monotone ascending sequence of simple L-functions

$$f_{11}, f_{22}, f_{33}, \cdots$$

which, and also the related simple U-functions, tend to f; they therefore define the integral of f.

Now this sequence has all its terms after the mth greater than the corresponding terms of the auxiliary sequence of f_m. Hence

$$\int_a^b f\,dg \geqslant \int_a^b f_m dg \text{ for all } m.$$

But we also have, for all m,

$$\int_a^b f_m dg \geqslant \int_a^b f_{mm} dg.$$

The theorem follows by making $m \to \infty$.

In particular the integral of an L-function bounded below is the limit of the integrals of any monotone ascending sequence of simple L-functions of which it is the limiting function.

Remembering that change of sign transfers a type to the twin type, we see that similarly we may define the *integral of any U-function bounded above** as the limit of the integrals of any monotone descending sequence of simple U-functions which has it as limiting function. Also we have the theorem that every monotone descending sequence of U-functions is integrable term by term.

The properties of the integrals of functions constant in stretches are at once extensible to those of semicontinuous functions.

I. If C is any constant, then $\int Cfdg = C\int fdg$.

For the generating simple functions of $C.f$ are C times those of f.

II. The integral of the sum of two functions is equal to the sum of their integrals.

This presupposes that the sum is semicontinuous and that the sum of the integrals has a meaning.

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* In the case of functions of both types, continuous functions, we have already seen that the two definitions coincide; it is also easily seen independently, as the difference of the two sequences is a monotone descending sequence of simple U-functions tending to zero, which as we know is integrable term by term.

If both functions are L-functions, their sum is the limit of the sum of their auxiliary sequences. The other cases reduce to this, because $-L=U$.

III. If f_1 is greater than or equal to f_2 throughout (a, b), then

$$\int_a^b f_1 dg \geqslant \int_a^b f_2 dg.$$

If both are L-functions we can so arrange that the generating functions of the latter are all less than or equal to the corresponding ones of the former.

If one is an L-function and the other a U-function, the difference of their generating sequences tends to a nowhere negative limiting function, which can also be generated by nowhere negative simple functions.

IV. If the interval be divided up in any manner into stretches the integral over that interval is the sum of the integrals over the partial stretches.

It follows immediately from (III) that the integrals of the simple U-functions of paragraph 2 (which lie between the corresponding simple L-functions and the limiting L-function) tend to that of the limiting function. From this we deduce that, *given any L-function bounded below, its integral is equal to the upper bound of the integrals of the U-functions bounded above nowhere greater than it.*

CHAPTER V

THE INTEGRATION OF FUNCTIONS (*continued*)

§ 1. **Final extension of integration.** Besides not being applicable to simple functions, Riemann's definition has the great disadvantage that it will in general fail in the case of the limiting function of a sequence whose terms are integrable according to his definition.

It is a remarkable fact that both these drawbacks disappear if we apply the very same method at the next stage.

DEFINITION. Given any function $f(x)$ whatever, bounded or unbounded, the upper bound of the integrals of the U-functions bounded above which nowhere exceed $f(x)$ is called the *lower integral* of $f(x)$; the lower bound of the integrals of all the L-functions bounded below which are nowhere less than $f(x)$ is called the *upper integral* of $f(x)$*.

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* N.B. If we permute the letters U and L we get Darboux's lower and upper evaluations.

DEFINITION. If its upper and lower integrals are equal, $f(x)$ is said to *possess an integral* (finite or infinite) which is their common value. If the integral is finite, $f(x)$ is said to be *integrable.*

FUNDAMENTAL THEOREM. The limiting function of a monotone sequence of integrable functions possesses an integral (finite or infinite).

Let $f_1, f_2, \ldots, f_n, \ldots$ be the given sequence and let f be its limiting function.

It follows from our definition of integral that there exist, corresponding to each function f_n of the sequence, U-functions bounded above nowhere greater than f_n and L-functions bounded below nowhere less than f_n whose integrals differ from that of f_n by less than $e \, . \, 2^{-n-1}$. Let $A_n(x)$ and $B_n(x)$ be any two of these, such that

$$A_n \geqslant f_n \geqslant B_n \, .$$

Let us suppose the given sequence was monotone ascending. We assert that any pair of the sequences A_n and B_n can be replaced by a pair of monotone ascending sequences a_n and b_n, such that

$$a_n \geqslant f_n \geqslant b_n,$$

$$\int a_n(x)\, dg(x) - \int b_n(x)\, dg(x) \leqslant e.$$

We need only choose $a_1 = A_1$, $b_1 = B_1$ and then define by recurrence a_n to be the function which is equal at every point to the greater of a_{n-1} and A_n, and similarly b_n the greater of b_{n-1} and B_n. By the fundamental property (ii) of p. 9, a_n is an L-function bounded below and b_n a U-function bounded above.

Further, we have at every point

$$\begin{aligned} a_2(x) - b_2(x) \leqslant \text{ either } & A_1(x) - B_1(x) \\ \text{or } & A_2(x) - B_2(x). \\ \therefore \leqslant (A_1(x) - B_1(x)) &+ (A_2(x) - B_2(x)). \end{aligned}$$

Hence

$$\int a_2(x)\, dg(x) - \int b_2(x)\, dg(x) \leqslant \left\{ \begin{aligned} & \int A_1(x)\, dg(x) - \int B_1(x)\, dg(x) \\ + & \int A_2(x)\, dg(x) - \int B_2(x)\, dg(x), \end{aligned} \right.$$

$$\therefore \leqslant e \, . \, (\tfrac{1}{2} + \tfrac{1}{4}).$$

Similarly

$$a_n(x) - b_n(x) \leqslant \{a_{n-1}(x) - b_{n-1}(x)\} + \{A_n(x) - B_n(x)\}$$

and $$\int a_n(x)\,dg(x) - \int b_n(x)\,dg(x) \leqslant e\,.\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}\right)$$
$$\leqslant e.$$

Now the limiting function of the sequence a_n is an L-function bounded below nowhere less than f and whose integral we have seen to be equal to the limit of those of its generating functions. Also $\int b_n dg$ remains always less than or equal to the upper bound of the integrals of the U-functions bounded below nowhere greater than f.

It follows immediately since e is arbitrary that the theorem is true.

THEOREM. A monotone sequence of integrable functions may be integrated term by term.

This follows immediately from the preceding argument since both the limit of the integrals and also the integral of the limiting function lie between the limit of $\int a_n dg$ and that of $\int b_n dg$, and since these limits differ by less than e.

THEOREM. A function obtainable by monotone sequences is either integrable or else both its upper and its lower integrals are infinite.

Suppose the lower integral is finite, there is then a U-function bounded above, having a finite integral, and nowhere greater than the given function f. It follows (from the second of the three fundamental properties of class) that the latter is obtainable by monotone sequences of functions greater than the U-function.

Now let $f_n(x)$ be the function which is at every point the lesser of the given function and of the constant n, this function is of the same type as f. Also it lies between the constant n and a certain integrable U-function; and we may therefore suppose this property possessed by all the functions of the system of monotone sequences defining f_n. It follows immediately by repeated application of the fundamental theorem, remembering that by the theorem just proved the integrals obtained at each stage are finite, that $f_n(x)$ is integrable.

Our given function therefore possesses an integral, by that same fundamental theorem. This integral cannot be infinite by hypothesis. The theorem follows.

For integrable functions we easily establish the four properties of pp. 20, 21. The proof is indeed but a repetition of the argument used in the case of semicontinuous functions.

(I) $\int Cf\,dg = C.\int f\,dg$,

(II) $\int (f_1+f_2)\,dg = \int f_1\,dg + \int f_2\,dg$,

(III) If $f_1 \geqslant f_2$ then $\int_a^b f_1\,dg \geqslant \int_a^b f_2\,dg$,

(IV) The $\int$ over an open interval = sum of $\int$'s over parts.

Let f be any integrable function (its integrability depends on the integrator; all functions are integrable with respect to a constant and have their integrals equal to zero).

Then f lies between two integrable semicontinuous functions, i.e. a U-function bounded above and an L-function bounded below. By adding suitable constants to them, we see that we may take the U-function to be negative and the L-function to be positive at every point, without affecting their integrability.

On the other hand it follows immediately from the last theorem of the preceding paragraph that *any function which lies between two integrable functions is itself integrable*, if it is one which can be obtained by monotone sequences.

This is in particular the case of the function equal to f, where f is positive, and zero elsewhere; and also of the function equal to f, where f is negative, and zero elsewhere. Changing the sign of the latter function and adding it to the other we get $|f|$. Consequently $|f|$ is integrable. *Thus if f is integrable so is $|f|$.*

For this reason a function which is integrable with respect to $g(x)$ according to our definition is said to possess an *absolutely convergent integral* with respect to $g(x)$.

Non-integrable functions fall into two classes: those possessing an integral (which is then infinite) and those whose upper integral is $+\infty$ and whose lower integral is $-\infty$.

For this second class of functions our definition fails entirely. It is however sometimes convenient to define for them a generalised integral with respect to $g(x)$. Under what conditions this is possible we do not propose to investigate here. That it is not always possible is easy to see; we need only take the case of a function which is $+\infty$ throughout one interval and $-\infty$ throughout another.

N.B. If D is a stretch

(1) over which the total increment of g vanishes, or

(2) throughout which f is zero,

then
$$\int_D f\,dg = 0$$
(for this is obvious by definition if f is constant in stretches; it is therefore true for semicontinuous functions and so on by induction for all functions mathematically definable).

From (2) and the fundamental property of integration II, it follows at once that assigning arbitrary values to a function on the boundary of the interval (a, b) will not affect its integral over that interval.

§ 2. Indefinite, parametric and repeated integration.

A function which is integrable in (a, b), is also integrable in every open interval contained in (a, b).

For let D be such an interval and f_1 the function equal to f in D and zero outside. Then
$$0 \leqslant |f_1| \leqslant |f|$$
at every point. Hence f_1 lies between the two integrable functions and, since it is obtainable by monotone sequences, it is integrable in (a, b).

Further,
$$\int_a^b f_1\,dg = \int_D f\,dg.$$

In particular, if ξ is any point of (a, b),
$$\int_a^\xi f\,dg = F(\xi), \text{ say,}$$
exists and is finite.

The function $F(x)$ is called the *indefinite integral* of f with respect to g. It is obvious that *if $f \geqslant 0$, F is monotone increasing.* For a simple increment of F is a sum of integrals over stretches and cannot be negative; a double increment of F is an increment of an increment, i.e. a simple increment of a sum of integrals, i.e. a sum of integrals, and therefore not negative. Similarly, all the increments of F simple, double, ..., $n$ple, are sums of integrals of f and consequently non-negative.

Since any integrable function can be regarded as the difference of two positive integrable functions, *an indefinite integral is the difference of two monotone increasing functions.*

We assert that, moreover, F is continuous at the endpoints of (a, b), i.e., that the unique limit of F in the (all +)-quadrant at a is zero, the unique limit in the (all −)-quadrant at b is $F(b)$.

Let a_n, b_n be any two points of the closed interval (a, b) such that (a_n, b_n) forms an interval, and let them describe two sequences of points such that

$$a_n \text{ lie inside } (a, a_{n-1}) \text{ and } a_n \to a$$
$$b_n \quad ,, \qquad ,, \quad (b_{n-1}, b) \text{ and } b_n \to b.$$

Let us suppose the integrand f is positive and let f be the function equal to f in (a, b) and zero elsewhere. By the remark at the end of § 1, we may clearly suppose f equal to zero outside the open interval (a, b), if it is even defined.

Then, as n increases, f_n forms a monotone sequence of integrable functions whose limit is f. Therefore

$$\int_a^b f_n dg \to \int_a^b f dg,$$

i.e.

$$\int_{a_n}^{b_n} f dg \to \int_a^b f dg.$$

This proves our statement in this case.

If f is the difference of two positive integrable functions, the result now follows at once.

It immediately follows that the indefinite integral F of f with respect to g has at most the discontinuities of $g(x)$. If $f = 0$ where g is discontinuous, F is continuous everywhere.

An indefinite integral is thus in our scheme of functions a very simple function indeed.

Let us now suppose that instead of one function f and one function g, we are considering at every point x *a set of pairs of functions* $f(x)$ and $g(x)$, the f's being all obtainable by monotone sequences and the g's being further all monotone increasing functions. This set might be, for example, the succession

$$f_1, g_1; \ f_2, g_2; \ \dots; \ f_n, g_n; \ \dots$$

We might then suppose that, besides the coordinates which represent the point x, we have a further coordinate p representing the index n, for example, its inverse $1/n$.

In this new space we should have the two functions

$$f(x, p), \ g(x, p)$$

defined at all points (x, p) whose p-coordinate corresponds to an integral value of n, and we might, for example, agree to give to both the value zero elsewhere.

More generally, we may suppose that we are dealing with two functions

$$f(x, p), \ g(x, p),$$

of *two sets of variables* x and p, and that when p is fixed, g is monotone increasing with respect to x. If we now keep p fixed, and form the indefinite integral

$$\int^x f dg,$$

we shall have a new function of the two sets of variables x and p,

$$F(x, p) = \int^x_{p \text{ fixed}} f(x, p)\, dg(x, p).$$

This process is called *parametric integration.*

We may remark that $F(x, p)$ is obtainable by monotone sequences, for integration is a limiting process and we have seen that such a process performed on functions obtainable by monotone sequences, leads to the same kind of function. In fact, if $f(x, p)$ is obtainable by monotone sequences, starting with functions constant in stretches, F is so too starting with functions constant in stretches with respect to p and monotone increasing with respect to x, and these are themselves obtainable by monotone sequences starting with functions constant in stretches with respect to (x, p).

In particular, we may apply to F the process of integration with respect to a monotone increasing function of p (keeping x fixed). We then obtain what may be called a *repeated indefinite integral.*

While integrating with respect to x, we may also keep fixed the interval over which we integrate; for example, it might be the interval (a, b); we should then obtain by integration the same result as by substituting b for x in $F(x, p)$, i.e. $F(b, p)$.

More generally, we may suppose the interval a to x over which we integrate keeping p fixed, to depend on p. Integration would then lead to the same result as substituting for the variables x in F certain functions of p, i.e. we should get

$$F(x(p), p).$$

Integration with respect to a monotone increasing function of the variables p will then give us the most general kinds of *repeated definite integral.*

§ 3. Term-by-term integration of sequences and successions.

THEOREM I. If a monotone sequence of integrable functions whose limit is integrable be integrated term by term, then the integrated sequence converges uniformly to the indefinite integral of the limiting function.

(In the case when the integrator g is continuous, this is a corollary of Appendix, 1, p. 46.)

Case 1. The functions are all positive. Then

$$\left|\int_a^x f_n\,dg - \int_a^x f\,dg\right| \leqslant \left|\int_a^b f_n\,dg - \int_a^b f\,dg\right|.$$

We can find an n from and after which the righthand side is less than e; the same n makes the lefthand side less than e.

Case 2. General case.

$$f_n = a_n - b_n;\ f = a - b,$$

where a_n, b_n, a, b, are positive functions and we can so arrange that a_n tends to a, b_n to b.

By case 1, we can find an n from and after which both

$$\left|\int_a^x a_n\,dg - \int_a^x a\,dg\right| \quad\text{and}\quad \left|\int_a^x b_n\,dg - \int_a^x b\,dg\right|$$

are less than $e/2$. Then, from and after that n,

$$\begin{aligned}\left|\int_a^x f_n\,dg - \int_a^x f\,dg\right| &= \left|\left(\int_a^x a_n\,dg - \int_a^x a\,dg\right) - \left(\int_a^x b_n\,dg - \int_a^x b\,dg\right)\right| \\ &\leqslant \left|\int_a^x a_n\,dg - \int_a^x a\,dg\right| + \left|\int_a^x b_n\,dg - \int_a^x b\,dg\right| \\ &< e.\end{aligned}$$

This proves the theorem.

Theorem II. If a sequence of functions whose bounding functions are integrable be integrated term by term, the integrated sequence converges uniformly to the integral of the limiting function.

(This contains the preceding as a particular case.)

f is the limit of a monotone ascending sequence a_n, and of a monotone descending sequence b_n, of integrable functions, such that

$$a_n \leqslant f_n \leqslant b_n$$

at every point. Determine N so that both

$$\left(\int_a^x a_n\,dg - \int_a^x f\,dg\right) \quad\text{and}\quad \left(\int_a^x b_n\,dg - \int_a^x f\,dg\right)$$

are less than e in absolute value.

$$\int_a^x f_n dg - \int_a^x f\,dg$$

lies between them, i.e., a fortiori, is less than e in absolute value. Q.E.D.

THEOREM III. If a succession of functions whose bounding functions are integrable, and whose upper and lower functions have the same integral be integrated term by term, the integrated succession will cease to oscillate, and will converge uniformly to the common value of the integrals of the upper and lower functions.

(This again contains both the preceding theorems as particular cases.)

We have
$$\int_a^x a_n\,dg \leqslant \int_a^x f_n\,dg \leqslant \int_a^x b_n\,dg$$
and both the extreme left and the extreme righthand side converge uniformly to their common limit. Q.E.D.

THEOREM IV. Given a succession of functions f_n such that
$$\int f_n\,dg \to \int f\,dg$$
uniformly in every interval, if, further, f_n is in absolute value less than a fixed constant, and ϕ is any integrable function, then
$$\int \phi f_n\,dg \to \int \phi f\,dg \text{ uniformly.}$$

We first show that if ϕ is any integrable function, there are functions constant in stretches ϕ_m such that
$$\int_a^b \left|\phi - \phi_m\right| dg < \frac{1}{m}.$$

From the integrability of ϕ, we deduce the existence of an *L*-function ϕ_m' bounded below and nowhere less than ϕ, such that
$$\int_a^b (\phi_m' - \phi)\,dg < \frac{1}{2m}.$$

This *L*-function is the limit of an ascending sequence of simple *L*-functions. There therefore exists a simple *L*-function ϕ_m nowhere greater than ϕ_m' and such that
$$\int_a^b (\phi_m' - \phi_m)\,dg < \frac{1}{2m}.$$

But
$$\begin{aligned}|\phi - \phi_m| &\leqslant |\phi - \phi_m'| + |\phi_m' - \phi_m| \\ &= (\phi_m' - \phi) + (\phi_m' - \phi_m).\end{aligned}$$

Hence
$$\int_a^b \left|\phi - \phi_m\right| dg < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}.$$

The theorem to be proved now follows at once. The absolute values of f_n are less than a fixed constant M, say. Now
$$\int \phi f_n\,dg = \int \phi_m f_n\,dg - \int f_n(\phi_m - \phi)\,dg;$$

therefore

$$\int \phi f_n \, dg - \int \phi f \, dg = \int \phi_m (f_n - f) \, dg$$
$$+ \int f(\phi_m - \phi) - \int f_n (\phi_m - \phi) \, dg.$$

Since ϕ_m is constant in stretches, the first integral on the right tends uniformly to zero if m is fixed. The absolute value of each of the two last integrals is less than M/m, a quantity which can be made as small as we please by making m large enough. The theorem follows at once.

N.B. The fact that the succession was bounded was essential to the proof. In the next theorem, this is replaced by a more general condition; but ϕ is bounded.

Theorem V. If

$$\int f_n \, dg \rightarrow \int f \, dg$$

uniformly in every interval; if, further, $|f_n|$ remains less than an integrable function ψ, and if ϕ is any bounded function, then

$$\int \phi f_n \, dg \rightarrow \int \phi f \, dg \text{ uniformly.}$$

Let $\psi_{(M)}$ be the function which is equal to ψ where ψ is greater than the constant M, and which is equal to M elsewhere. We first prove that we can always so determine M that

$$\int (\psi_{(M)} - M) \, dg < e.$$

The functions $\{\psi + M - \psi_{(M)}\}$ form, as M increases to infinity, a monotone ascending sequence of functions whose limit is ψ, for each of them is equal to ψ where ψ is $< M$, and equal to M elsewhere. We can therefore determine M so that

$$\int \{\psi + M - \psi_{(M)}\} \, dg > \int \psi \, dg - e$$

by the theorem of term by term integration of a monotone sequence. M so determined will do.

By the fundamental property of integration III, we then have, a fortiori, for all n,

$$\int (|f_n|_{(M)} - M) \, dg < e.$$

It is clearly no restriction to suppose that, also

$$\int (|f|_{(M)} - M) \, dg < e,$$

which certainly holds if $|f|$ is smaller than ψ. We now proceed as in the last theorem. Let B be the maximum of ϕ, and let ϕ_m be a function constant in stretches nowhere greater than B in absolute value, such that

$$\int \left| \phi - \phi_m \right| dg < \frac{e}{M}.$$

Then, as before,

$$\int \phi f_n \, dg - \int \phi f \, dg = \int f (\phi_m - \phi) \, dg - \int f_n (\phi_m - \phi) \, dg + \int \phi_m (f_n - f) \, dg \quad \text{......(1).}$$

Now
$$\left| \int f_n (\phi_m - \phi) \, dg \right| \leqslant \int |f_n| \left| \phi_m - \phi \right| dg$$
$$\leqslant \int |f_n|_{(M)} \cdot \left| \phi_m - \phi \right| dg$$
$$\leqslant M \int \left| \phi_m - \phi \right| dg + 2B \int (|f_n|_{(M)} - M) \, dg$$
$$< e \, . \, (1 + 2B).$$

Similarly,
$$\left| \int f (\phi_m - \phi) \, dg \right| < e \, . \, (1 + 2B).$$

Further, since ϕ_m is constant in stretches, we have uniformly,

$$\underset{n=\infty}{Lt} \int \phi_m (f_n - f) \, dg = 0.$$

Hence, since e is arbitrary, the lefthand side of (1) differs as little as we please from zero. Q.E.D.

§ 4. Theorems concerning the integrator.

It is convenient to define integration with respect to a function *which is the sum of two monotone functions* by means of the formula

$$\int f \, d(g_1 + g_2) = \int f \, dg_1 + \int f \, dg_2.$$

This formula holds when $g_1 + g_2$ are monotone in the same sense: this is obvious when f is a constant and is therefore equally true when f is constant in stretches. Also if it is true of the generic term of a monotone sequence, it is true of the limiting function. (It is assumed that the righthand side has a meaning.)

It follows immediately from the validity of the formula in this case that when the functions g_1, g_2 are monotone in opposite senses, the righthand side remains unaltered when g_1 and g_2 are replaced by g_1', g_2'

where $g_1 + g_2 = g_1' + g_2'$. In fact we have, assuming g_1, g_1' monotone in the same sense,

$$\left\{\int f\,dg_1 + \int f\,dg_2\right\} - \left\{\int f\,dg_1' + \int f\,dg_2'\right\} = \int f\,d(g_1 - g_2') - \int f\,d(g_1' - g_2) = 0.$$

The definition is therefore consistent, that is to say, independent of the choice of g_1 and g_2, dependent only on their sum $(g_1 + g_2)$.

Theorems involving variation of the integrator are difficult unless they can be deduced from theorems involving only variation of the integrand. In this way we are able to prove the following theorems.

THEOREM I. If u, v and uv are integrable with respect to g, and

$$U = \int_a^x u\,dg, \quad V = \int_a^x v\,dg,$$

then

$$\int_a^x uv\,dg = \int_a^x u\,dV = \int_a^x v\,dU.$$

It is no restriction to suppose v *positive* integrable as it is the difference of two such functions.

Now if u is constant, the theorem is obvious. If u is constant in stretches, the theorem, true in each of these, is by the fundamental property IV true in their sum. Finally, if u is the limit of a monotone sequence u_n, uv is the limit of the *monotone* sequence $u_n \,.\, v$ (monotone, since v is positive). By the theorem on term by term integration of a monotone sequence, if the theorem is true for u_n, it is equally true for u. This proves the theorem by induction for all functions mathematically definable.

THEOREM II. If the integrator g is a function of the two sets of variables x and y, while f depends on the variables x only, then

$$\int_{00}^{ab} f\,dg = \int_0^a f\,d\left\{\int_0^b \underset{x \text{ is fixed}}{dg}\right\}.$$

This is proved in exactly the same way as the preceding theorem. It is obvious for a constant, hence true for a simple function, and by the theorem on term by term integration of a monotone sequence follows in general.

THEOREM III. If for a constant t, $g(x, t)$ is the difference of two monotone increasing functions of x, and if $h(t)$ is the difference of two monotone increasing functions of t, then

$$\int^t dh(t)_{x \text{ fixed}} \left\{\int^x f(x)\,d\underset{t \text{ fixed}}{g(x, t)}\right\} = \int^x f(x)\,d\left\{\int^t \underset{x \text{ fixed}}{g(x, t)}\,dh(t)\right\}.$$

It is again sufficient to prove this for the case when $f(x)$ is constant, that is to say it suffices to prove that

$$\int^t dh(t)_{x\text{ fixed}} \left\{\int^x dg\underset{t\text{ fixed}}{(x,\, t)}\right\} = \int^x d_{t\text{ fixed}} \left\{\int^t g\underset{x\text{ fixed}}{(x,\, t)}\, dh(t)\right\}.$$

Now, if x is one single variable,

$$\int_a^x dg\underset{t\text{ fixed}}{(x,\, t)} = g(x,\, t) - g(a,\, t).$$

If we integrate with respect to $h(t)$, we get, on the right,

$$\int g\underset{x\text{ fixed}}{(x,\, t)}\, dh(t) - \int g(a,\, t)\, dh(t).$$

If x stands for two variables, say ξ and η, then

$$\int_a^x dg\underset{t\text{ fixed}}{(x,\, t)} = \begin{bmatrix} g\{(\xi-0,\, \eta-0),\, t\} + g\{(\alpha+0,\, \beta+0),\, t\} \\ -g\{(\xi-0,\, \beta+0),\, t\} - g\{(\alpha+0,\, \eta-0)\ \ t\} \end{bmatrix}$$

and if we integrate with respect to $h(t)$ we again get the same as

$$\int^x d_{t\text{ fixed}} \left\{\int^t g\underset{x\text{ fixed}}{(x,\, t)}\, dh(t)\right\}.$$

Similarly, the formula holds if x stands for any number of variables. This proves the theorem when $f(x)$ is a constant. It follows at once that it is true for f constant in stretches. Now let

$$f_1,\, f_2,\, \ldots,\, f_n,\, \ldots \rightarrow f$$

be a monotone sequence of functions, converging to f. Then

$$\int f\, dG = \text{Lim} \int f_n\, dG$$

for every G, difference of two monotone increasing functions. In particular

$$q(t) = \int f(x)\, dg\underset{t\text{ fixed}}{(x,\, t)} = \text{Lim}\ q_n(t) = \int f_n(x)\, dg\underset{t\text{ fixed}}{(x,\, t)}.$$

Also, $q_n(t)$ forms as n increases a monotone sequence (g being taken to be monotone increasing, for brevity). Therefore,

$$\int q(t)\, dh(t) = \text{Lim} \int q_n(t)\, dh(t).$$

But since f_n may be integrated term by term with respect to

$$\left[\int^t g\underset{x\text{ fixed}}{(x,\, t)}\, dh(t)\right],$$

it follows that if the theorem holds for f_n, it does also for f.

THEOREM IV.

$$\int^y \left\{\int^x f\underset{y\text{ fixed}}{(x,\, y)}\, dg(x)\right\} dh(y) = \int^{xy} f(x,\, y)\, d\left\{g(x) \,.\, h(y)\right\}.$$

The proof is so similar to the preceding ones that it may be left as an exercise to the student.

APPENDIX

1. *Continuity and uniform convergence.*

We saw that a function is said to be continuous at a point where it is both upper and lower semicontinuous and assumes a finite value. Hence, if $f(x)$ is continuous at a point x_0, there is an interval surrounding x_0 throughout which

$$|f(x)-f(x_0)| < e.$$

This is the e-definition of continuity at a point, and is due to *Cauchy*. Furthermore, if we are given any sequence of points contained in the interval of definition, and converging to x_0 as limiting point, the corresponding sequence of values of f will have $f(x_0)$ as unique limit. This is *Heine's* definition of continuity. The three definitions of continuity are absolutely equivalent, provided in the case of the third we add the condition that $f(x_0)$ is finite. It is clear that both Cauchy's and Heine's definition can be deduced from ours. $f(x_0)$ is the unique limit of the upper and of the lower bounds of f in the neighbourhood of x_0. We can therefore determine an interval containing x_0 in which the difference between the upper and lower bounds of f is less than e. Throughout that interval, we shall have, a fortiori,

$$|f(x)-f(x_0)| < e.$$

On the other hand, if we are given a sequence of points

$$x_1 \dots x_n \dots$$

converging to x_0 as limiting point, then the corresponding sequence of values of f lies between the two sequences of the upper and lower bounds of f in the intervals

$$(x_1 x_0) \dots (x_n x_0) \dots$$

which shrink up to x_0.

Conversely our definition may be deduced from any of the two others. If we have determined an interval surrounding x_0 throughout which

$$|f(x)-f(x_0)| < \tfrac{1}{2}e,$$

then the same inequality holds if we replace $f(x)$ by its upper or its lower bound; adding the two inequalities thus obtained, it follows that these bounds differ by less than e.

On the other hand, if Heine's definition is fulfilled, consider any sequence of intervals shrinking up to x,

$$w_1, w_2, w_3, \dots.$$

Let x_1 be any point of w_1, where f assumes a value differing from its upper bound by less than e, x_n any point of w_n where f_n assumes a value differing by less than e/n from its upper bound in w_n; then the points[*]

$$x_1, x_2, x_3, \dots$$

---

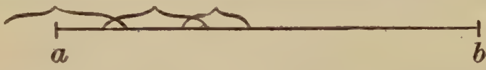
[*] This involves the so-called Zermelo axiom. A similar argument was used to prove the fundamental theorem of integration on p. 30 but might have been avoided.

form a sequence converging to x_0. Hence the limit of the upper bounds, which coincides with the limit of $f(x_n)$, is $f(x_0)$. A similar argument applies to the lower bounds.

A function is said to be continuous throughout an interval if it is continuous at every point of that interval. In order to investigate the properties of a continuous function it is convenient to prove the following lemma, from the theory of sets of points.

The Heine-Borel theorem. If to each point x of a closed interval (ab) we make correspond an interval containing that point as internal point, then a finite number of these intervals cover the whole of (a, b).

Let us prove it first for a one-dimensional interval.



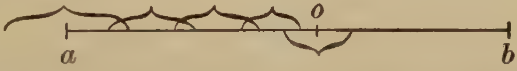
Let d_1 be the interval which corresponds to a and let x_1 be its second endpoint (the first endpoint lies outside (a, b)); let d_2 be the interval corresponding to x_1 and let x_2 be its second endpoint and so on.

Each of the points x_n so obtained has the property that all the points which precede it are internal to a finite number of the given intervals.

We may therefore divide the points of (a, b) into two classes:

(1) the points x such that every interval interior to (a, x) can be covered by a finite number of our intervals;

(2) the points not possessing this property. We shall prove that the points of this second class do not exist.



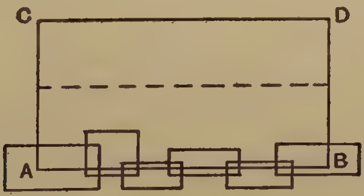
Let O be the last point of the first class; this point clearly exists since if all the points preceding O are of the class (1) so is O.

Now all the points internal to the interval which corresponds to O clearly belong to the first class.

Hence unless O is the last point of (a, b), O cannot have been the last point of the first class.

This proves the theorem.

Let us now prove it for two dimensions. Let $ABCD$ be the given rectangle. By the one-dimensional theorem all the points of AB may be covered by a



finite number of the intervals; let h_1 be the smallest distance from AB of the horizontal sides of these intervals. Then all the points whose height above (a, b) is less than h_1 are internal to a finite number of our intervals.

Let h be the largest number such that all the points of height less than h above AB and which belong to our rectangle are internal to a finite number of our intervals.

The argument of the one-dimensional theorem proves that h is not less than the height above AB of CD.

This proves the theorem in two dimensions and similarly in n.

Uniform continuity. If $f(x)$ is continuous in a closed interval of definition, then given any positive number e, there exists a positive number d, such that if x_1 and x_2 are any two points of the interval of definition distant less than d, then
$$|f(x_1)-f(x_2)| < e.$$

For by the theorem just established, if to every point x_0 we make correspond an interval i surrounding it, such that throughout the concentric interval I twice as large in every direction, we have
$$|f(x)-f(x_0)| < e/2,$$
a finite number of the intervals i cover the whole interval of definition.

Let d be half of the smallest side of these intervals. Then if x_1 is any point of E, x_1 is internal to at least one interval, i_2 say, of the finite set of intervals. The interval I_1 twice as large as i_1 therefore contains all the points distant less than d from x_1, in particular x_2.

If x_0 is the point to which I_1 corresponds, then by adding the inequalities got by writing in turn x_1 and x_2 for x, the theorem follows.

Closely allied to the idea of continuity of a function is that of *uniform convergence of a sequence of functions**.

Let $f_1, f_2, \dots, f_n, \dots$ be any sequence of functions of x†, and let f be their limiting function.

We suppose all these functions bounded and defined in the same interval.

Let us imagine that, besides the coordinates which represent the point x, we have a further coordinate p representing the parameter n, for example its inverse $1/n$.

In this new space we define a new function
$$F(x, p)=f_n(x)-f(x),$$
but only at those points where p represents an integral value of n, and let us agree to give it the value 0 elsewhere.

Then the convergence of the given sequence is said to be uniform at a point x_0, if in our new space the upper and the lower limits of the function $F(x, p)$ at the point which corresponds to $x=x_0$ and $n=\infty$ coincide and are equal to 0; in other words, F is continuous at that point.

This may also be expressed by saying that there is *an unique* $\overline{m+1}$*'ple limit* at x_0, where m is the number of coordinates of x.

---

* It has been better termed by Du Bois-Reymond *continuous convergence* ("stetige Convergenz").

† Not necessarily continuous.

In the e-language this may be stated as follows:

The given sequence is said to converge uniformly at the point x_0 if, given any positive quantity e, we can find an index n_0 and an interval surrounding x_0, such that throughout it and from and after $n=n_0$ we have

$$|f_n(x)-f(x)|<e.$$

The notion of uniform convergence at a point is of great importance in the theory of functions of real variables; moreover, many properties laboriously deduced from the e-definition in a closed interval become obvious by inspection.

When the convergence is uniform at every point of a given interval (open or closed), it is said to be uniform in that interval. The equivalence of this definition with that given in Chapter I when the interval is closed is easily proved with the help of the Heine-Borel theorem; and it may be shown that we can also define uniform convergence in an open interval as uniform convergence in every closed component.

It is obvious that, if we can determine n_0 such that throughout the closed interval (a, b) and for all n after n_0 we have

$$|f_n(x)-f(x)|<e,$$

then the convergence is uniform at every point of (a, b).

Conversely, if the convergence is uniform at every point of (a, b) then, given e, we can at each point find an n_0 and an interval containing the point, throughout which from and after n_0, we have the above inequality. A finite number of these intervals cover the whole of (a, b); if n_1 is the greatest of the corresponding n_0's then throughout (a, b) from and after n_1 the above inequality holds. Q.E.D.

THEOREM. A function $f(x)$ which is the limit of a uniformly convergent sequence of functions of a certain type is also the limit of a monotone ascending and of a monotone descending sequence of functions of that type.

It is sufficient to prove it for the case in which the functions are all defined in a closed interval (a, b).

Let $e_1, e_2, \ldots, e_n, \ldots$ be a monotone sequence of positive numbers tending to zero.

Let f_1 be the first of our functions which throughout (a, b) fulfils the inequality

$$|f_1(x)-f(x)|<e_1.$$

Similarly, let f_n be the first to satisfy the inequality

$$|f_n(x)-f(x)|<e_n$$

throughout (a, b).

The functions f_n+e_n and f_n-e_n both belong to the same type as f_n and the corresponding sequences are respectively monotone ascending and monotone descending and they tend to $f(x)$. This proves the theorem.

COROLLARY. $f(x)$ belongs to the same type.

If the generating functions are continuous it follows that so is the limiting function. The connection between continuity and uniform convergence is also illustrated by the following theorem.

THEOREM. If a *monotone* sequence of continuous functions has as limit a continuous function, it converges uniformly.

Again take the interval to be closed. Let f_n be the generic term of the sequence and f the limiting function. Suppose the sequence monotone descending. The differences

$$f_n - f$$

form a monotone descending sequence of continuous functions, which are particular cases of L-functions.

By the theorem of bounds the limit of their upper bounds is zero. We can therefore find an n_0, from and after which, it is less than e. Therefore, a fortiori, throughout (a, b)

$$f_n - f < \epsilon \text{ for } n > n_0.$$

Similarly if the sequence is monotone ascending.

2. *The generation of sets of points, and the theory of content.*

A set of points is said to be well-defined if we are given a law which enables us to say of any arbitrarily chosen point whether it belongs to the set or not. If we are given such a set E, we are at once able to define the function which is 1 at every point of the set and 0 elsewhere. Conversely, given any function which only assumes the two values 0 and 1, then the set of points where this function is 1, is well defined. This enables us to generate all well-defined sets of points.

The simplest functions constant in stretches which are not actual constants are those which only assume two values, say, 0 and 1. If such a function is a simple U, the set of points where it is 1 consists of a finite number of closed intervals. We shall call it a simple U-set. The complementary set is a simple L-set.

If we are given a monotone descending sequence of simple U-functions, all of which assume only the values 0 and 1,

$$E_1(x),\ E_2(x),\ \ldots\ E_n(x),\ \ldots,$$

then the set of points E_n, where $E_n(x)$ is unity, is clearly contained in E_{n-1}. We have thus got what we may call *a monotone descending sequence of sets of points*,

$$E_1 > E_2 > \ldots \quad \text{(The symbol $>$ now means containing.)}$$

The set of points E common to all of them will be called *their limiting set* of points, and is identical with the set of points where the function $E(x)$, limit of $E_n(x)$, is unity. Since $E(x)$ is a U-function which assumes only the values 0 and 1, the set of points E will be called a (general) U-set. The complementary set, limit of monotone ascending sequence of simple L-sets (the limiting set now consists of all the points belonging to at least one of the generating sets) is, similarly, a (general) L-set. In this way, proceeding stage by stage as in the generation of functions, we obtain the most general kind of set of points mathematically definable.

Given any set of points E, and a monotone increasing function $g(x)$, the integral of the function $E(x)$ which is 1 in E and 0 elsewhere, may be called the *total variation* of $g(x)$ with respect to E, or, as we shall say for brevity, *the* $g(x)$*-content* of E.

In order to evaluate the $g(x)$ content of E, we have to find the upper bound of the integrals of the U-functions less than $E(x)$. It is easily seen that it is sufficient to take into consideration those U-functions only, which assume the values 0 and 1 and no others. It is evident that we need only consider non-negative U-functions. Let $a(x)$ be such a function; where $a(x)$ is greater than or equal to $1/n$, we put it equal to 1. This does not affect its upper semicontinuity, nor its property of being nowhere greater than $E(x)$. If we put it equal to 0 where it is less than $1/n$, and make n tend to infinity, we obtain a monotone sequence of U-functions assuming only the values 0 and 1, whose integrals tend to a value greater than the integral of $a(x)$.

It follows immediately that *the content of E is the upper bound of the content of U-sets contained in E.*

We remark that *a U-set is identical with a closed set.* A U-set is closed; this in fact is almost obvious. Let $E(x)$ be the function which is 1 in E and zero elsewhere. Then at a limiting point of E, the function $E(x)$ cannot assume a value less than 1 since it is upper semicontinuous. Conversely, every closed set is a U-set*. For let E be the closed set and let $E(x)$ be 1 in E, 0 outside. Then $E(x)$ is upper semicontinuous. For a point where $E(x)$ is zero is internal to an interval containing no point of E (if such an interval did not exist the point would be a limiting point of E, i.e. $E(x)$ would be 1); therefore at such a point, the upper and lower limits of $E(x)$ are both zero. On the other hand, at a point where $E(x)$ is 1, $E(x)$ is obviously not less than the upper and lower limits. Hence $E(x)$ is upper semicontinuous and E is a U-set.

To denote the $g(x)$ content of E, we use the symbol $m_g(E)$. m stands for measure, which some writers use instead of content.

Historically, although, as we have seen, the general idea of the theory of integration follows naturally from the ideas of Cauchy and Darboux, the theory of content preceded the modern theory of integration. This was the main reason why mathematicians of the old school did not at once recognise its value.

We now proceed to prove some important theorems.

Theorem. If $f(x)$ is zero except at a set of points of g-content zero, the integral of f with respect to g vanishes.

Let E be the set of points where $f(x)$ is not zero, and $E(x)$ the function which is 1 throughout that set and zero elsewhere. Then, clearly, $f(x)$ equals $f(x) \times E(x)$. Therefore

$$\int f(x)\,dg = \int f(x)\,E(x)\,dg.$$

But†

$$\int f(x)\,E(x)\,dg = \int f(x)\,d\left\{\int E(x)\,dg\right\},$$

---

* This is equivalent to the following theorem of Cantor: A closed set consists of the points non-internal to a set of non-overlapping intervals.

† By Theorem I, p. 40.

or, since the integral of $E(x)$ is zero in every interval,

$$\int f dg = 0. \qquad \text{Q.E.D.}$$

THEOREM. If
$$\int f dg = 0$$
in every interval, then $f(x)$ is zero except, at most, at a set of g-content zero.

Since $\int f dg$ is the uniform limit of the integral of the function f_n identically equal to zero, and since f_n, being constantly zero, remains less than an integrable function, it follows from Theorem IV of p. 37 that the integral of $|f|$ is the uniform limit of that of $|f_n|$, that is, zero. We may therefore suppose f to be non-negative. Now, the set of points E_n where f is not less than $1/n$ has content zero because otherwise

$$0 = \int f dg \geqslant \frac{1}{n} \int E_n(x) \, dg > 0$$

by the fundamental property of integration III.

Now the set of points E where f is not zero is the limit of E_n as n tends to infinity, for it consists of all the points belonging to at least one E_n. By the theorem on term by term integration of a monotone sequence, the g-content of E is zero.

3. *Functions defined in a set of points.*

So far we have only considered functions which are defined at every point of an interval. More generally, we may suppose the values of a function to be given at some points, unknown at others. Such a function is *defined in a set of points.*

y is said to be function of x in a set of points E, if to each x belonging to that set there corresponds a single value of y.

A function which is defined in an interval is obviously also defined in every set of points contained in that interval.

In speaking of a function defined in a set of points E we merely exclude all consideration of other points, whether or not the function is defined at any of the latter.

A function defined in a set of points E has for upper bound in an interval D, the upper bound of those of its values which correspond to points of E inside the interval D.

With this convention, we may also define upper and lower limits of our function at every point x_0 such that in each interval containing x_0 there is a point of E (other than x_0): on the other hand, if there is an interval containing x_0 but no other point of E, then it is clearly impossible to define the upper and lower bounds of our function in that interval excluding x_0, and consequently impossible to define upper and lower limits at x_0 in this case. Hence the upper and lower limits of a function defined in a set of points E exist solely at points of the set E' of the limiting points of E; at any such point x_0 (which need not of course belong to E) they are defined to be the unique limits of the upper and lower bounds, respectively, of f in any succession of intervals each containing the point x_0 as internal point and shrinking up to x_0 (the point x_0 being, as before, excluded in the evaluation of the bounds in each interval).

Upper and lower semicontinuity. Continuity. A function is said to be upper semicontinuous at a point x_0 where it is defined, provided its upper limit at that point (if existent) is not greater than its value at that point.

A function is said to be continuous at a point if it is both upper and lower semicontinuous at that point.

Thus continuity at a point depends on the set of points at which the function is defined. A function, defined in E, may be discontinuous at a point x_0 of E and yet become continuous at x_0 when considered to be defined in a subset E_1 of E.

A function is said to be *continuous over a set of points* E, if, when considered as defined only in that set E, it is continuous at every point of E.

A function is said, *at a point* x_0, to be *continuous with respect to a set of points* E, if, when considered as defined only in that set E, it is continuous at x_0. Obviously a function which is continuous over a set of points E is continuous over every component of E.

This may also be expressed in the e-language: f is continuous over E if, to every point x_0 of E, we can make correspond an interval surrounding x_0, throughout which, for all points x of E, we have

$$|f(x)-f(x_0)| < e.$$

The Generalised Heine-Borel Theorem. If to each point x of a closed set of points E we make correspond an interval containing that point as internal point, then a finite number of these intervals cover the whole set E.

Let (a, b) be an interval containing E, and let E' be those of its points which do not belong to E. Let (a, b) be closed. To every point of E' we can make correspond an interval surrounding it containing no points of E.

To each point of (a, b) now corresponds an interval. A finite number of these intervals cover the whole of (a, b). Of these intervals those corresponding to points of E' contain no points of E; the others cover the whole of E.

Q.E.D.

Uniform continuity. If $f(x)$ is continuous over a closed set of points E, then, given any positive number e, there exists a positive number d, such that, if x_1 and x_2 are any two points of E distant less than d, then

$$|f(x_1)-f(x_2)| < e.$$

The proof is but a repetition of the one on p. 44 *mutatis mutandis.*

THEOREM. If $f(x)$ is continuous over a *closed* set of points E, it assumes its upper and its lower bound in that set.

The argument is exactly the same as for the corresponding theorem for a semicontinuous function in an interval. The point constructed, being the limit of a set of intervals all of which contain points of E, certainly belongs to E since E is closed.

COROLLARY. An interval cannot be the sum of two closed sets of points without common points*.

The distance between two points is a continuous function of position and assumes its lower bound as the two points vary in the two sets. This lower bound cannot therefore be zero.

THEOREM. The set of points of E at which $f(x)$ assumes values not less (or not greater) than K is closed.

At a limiting point of such points, $f(x)$ by continuity assumes a value not less (not greater) than K.

COROLLARY. A function which is continuous in a closed interval† there assumes all values between its upper and its lower bound. For let K be any such a value, then the two closed sets where it is $\geqslant$ and where it is $\leqslant K$ must by the corollary above have common points.

Successions of functions.

Just as we considered in Chapter I successions of functions defined in an interval, we may consider successions of functions defined in a set of points E, and apply to them the investigation of Chapter I.

In the case when the set of points is closed, the convergence of the functions $f_n(x)$ in that set of points is said to be *uniform*, if, given any positive number e, an index N can be found, independent of x, such that, for every n from and after N,

$$|f_n(x)-f(x)|\leqslant e,$$

for every x belonging to the set of points.

More generally we may define uniform convergence as in Appendix, 1.

THEOREM. A sequence of functions, continuous over a closed set E, and which converges uniformly over E, converges to a function which is also continuous over E.

Proof:

$$|f(x_1)-f(x_2)|\leqslant|f_N(x_1)-f(x_1)|+|f_N(x_2)-f(x_2)|+|f_N(x_1)-f_N(x_2)|.$$

Choose N so that $|f_N(x)-f(x)|\leqslant e/3$ for all the x's of E, which we can, because of the uniform convergence, and δ so that $|f_N(x_1)-f_N(x_2)$ may be less than $e/3$ as soon as x_1 and x_2 are distant less than δ and belong to E; then

$$|f(x_1)-f(x_2)|\leqslant e.$$

The generation of functions defined in a set of points.

Here proceed as in Chapter II. The simplest functions defined in E are those which are constant in E. The next simplest are the functions constant in stretches, i.e. if (a, b) be any interval containing E, it is the sum of a finite number of stretches, inside each of which at all the points belonging to E the function is a finite constant.

---

[*] A set of points which is not expressible as the sum of two closed components without common points is called a continuous set of points (Jordan).

[†] More generally "over a continuous set of points."

Then as in Chapter II we generate by monotone sequences the most general kind of function it is possible to define in E. We can also shew that, if E is closed, a uniformly convergent sequence of functions of a certain type generates a function of the same type*.

We now come to a remarkable theorem (Egoroff).

A function defined in a set of points E and obtainable by monotone sequences, if finite throughout that set, is continuous over a closed subset C_n, whose g-content differs from that of E by less than $1/n$, n being an arbitrary positive integer.

This theorem is an immediate corollary to the following theorem:

THEOREM. A sequence of functions, defined in a set of points E, and having a finite limit, will converge uniformly in a closed set of points C_n whose g-content differs from that of E by less than $1/n$.

Let $f_n \to f$ in a set of points E where f assumes only finite values, and let e and d be any pair of positive numbers.

Let E_r denote the set of points at which

$$|f_{r+h} - f| < e$$

for all positive integral values of h.

Obviously, $$E_1 < E_2 < \ldots;$$

also, every point of E belongs to some E_r, since the succession f_n converges in E and f assumes only finite values in E. Therefore

$$E_1 < E_2 < \ldots \to E.$$

Hence by the theorem on term by term integration

$$m_g(E_n) \to m_g E.$$

Hence since $m_g(E)$ is finite we can determine N so that

$$m_g(E_N) > m_g(E) - d.$$

Then in E_N $$|f_{N+h} - f| < e$$

for all h.

Let us choose for e a value $1/p$, p being a positive integer, and let us choose for d a value $\epsilon/2p$, ϵ being a positive number independent of p.

Let the value of N obtained be denoted by N_p, supposing ϵ fixed for the present.

Let $\mathcal{E}_1$ be the set E_{N_1},

$\mathcal{E}_2$ the set common to E_{N_1}, E_{N_2}, which may be called their product $E_{N_1} . E_{N_2}$,

$\mathcal{E}_3$ the set common to E_{N_1}, E_{N_2}, E_{N_3},

and so on.

Clearly $$\mathcal{E}_1 > \mathcal{E}_2 > \mathcal{E}_3 \ldots \to \mathcal{E} \text{ say.}$$

Also $$m_g(E - \mathcal{E}_1) < \epsilon/2 \qquad < \epsilon$$

$$m_g(E - \mathcal{E}_2) \leqslant m_g(E - \mathcal{E}_1) + m_g(E - E_{N_2}) \leqslant \epsilon\left(\tfrac{1}{2} + \tfrac{1}{4}\right) \qquad < \epsilon$$

$$m_g(E - \mathcal{E}_{n+1}) \leqslant m_g(E - \mathcal{E}_n) + m_g(E - E_{N_{n+1}}) < \epsilon\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}}\right) <$$

---

* This of course contains the theorem of the preceding page as a particular case.

But by the theorem on term by term integration

$$m_g(\mathcal{E}_n) \to m_g(\mathcal{E}).$$

But $\mathcal{E}$ belongs to E_{N_p} whatever p. In $\mathcal{E}$, therefore,

$$|f_{N_p+h} - f| < \frac{1}{p},$$

whatever p.

Hence the sequence f_n converges uniformly on a set of points $\mathcal{E}$ whose g-content differs from that of E by less than ϵ.

Now the content of $\mathcal{E}$ is the upper bound of that of its closed components. Hence there is a closed set C whose g-content differs from that of E by less than 2ϵ, in which the sequence f_n converges uniformly to its limiting function.

Q.E.D.

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BY

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INTEGRAL BASES

BY

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PREFACE

THE pages which follow give the result of an attempt to obtain the modulus of complex integers in the field of algebraic numbers defined by $\theta^n - a = 0$. Twenty-three cases require discussion and are considered in Chap. XV. The solution of some of the problems arising in this investigation led the writer to evolve various new methods which are applicable to the number-field defined by any irreducible equation

$$a(\theta) \equiv \theta^n + a_1\theta^{n-1} + \dots + a_n = 0.$$

After the disposal of preliminaries the new methods are discussed in Chaps. VI—VIII and an integral basis of the field defined by $a(\theta) = 0$ is obtained in Chap. IX. The rule for putting down the integral basis follows a process of approximation explained in Chaps. V—VIII, the first three stages of it being worked there. Failing cases exist but the approximations given are sufficient to cover nearly any numerical equation not specially constructed to defy them. The bases of prime ideals can also be put down in fields covered by the three approximations.

One more stage in the approximation (fourth dissection) has been worked in manuscript by the author but, as it appears to lead no nearer to finality than the ones given, it was not thought necessary to print this.

The methods of Chaps. V—X apply equally to a relative-field, a fact used in certain cases of $[\sqrt[n]{a}]$.

To make the investigation as elementary as possible an account of Dedekind's theory of algebraic numbers has been added in Chap. II.

A variety of causes has led to some delay in publication.

W. E. H. B.

1926

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CHAPTER I

INTRODUCTION AND SUMMARY

1. The *complex* (or *algebraic*) *integers* in a field of algebraic numbers are known to form an aggregate of the type

$$x_0\omega_0 + x_1\omega_1 + \dots + x_{n-1}\omega_{n-1},$$

n being the degree of the field, for rational integral values of $x_0, x_1, \dots x_{n-1}$, provided $\omega_0, \omega_1, \dots \omega_{n-1}$ is a suitably chosen set of complex integers of the field. The n integers $\omega_0, \omega_1, \dots \omega_{n-1}$, which form an *integral basis* (*Minimalbasis*) of the field are not unique but every two integral bases

$$(\omega_0, \omega_1, \dots \omega_{n-1}), \quad (\omega'_0, \omega'_1, \dots \omega'_{n-1}),$$

are connected by relations of the type

$$\omega'_i = p_{0i}\omega_0 + p_{1i}\omega_1 + \dots + p_{n-1\,i}\omega_{n-1},$$
$$\omega_j = q_{j0}\omega_0 + q_{j1}\omega_1 + \dots + q_{j\,n-1}\omega_{n-1},$$

where $|p_{ji}|$, $|q_{ji}|$, are reciprocal integral unit determinants.

A quadratic field can always be defined by the equation $\theta^2 - m = 0$, where m is an integer divisible by no square factor, and then an integral basis is known to be

$$\mathfrak{o} = (1, \tfrac{1}{2} + \tfrac{1}{2}\theta) \quad \text{or} \quad (1, \theta)$$

according as

$$m \equiv 1 \pmod 4) \quad \text{or} \quad m \equiv 2, 3 \pmod 4).$$

Methods of finding an integral basis of a cubic field are due independently to G. B. Mathews and G. T. Woronoj. An integral basis of a cyclotomic field has been obtained by Kummer, while Hilbert's detailed investigations of quadratico-quadratic fields include a discussion of the algebraic integers therein. Little appears to have been done beyond these cases.

2. In the following pages I develop a method of finding all the algebraic integers of the field $[\theta]$ defined by a rational integral irreducible equation

$$a(\theta) \equiv \theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + \dots + a_n = 0.$$

The field has an integral basis of the form

$$\mathfrak{o} \equiv \left(1, \frac{\psi_1(\theta)}{\Delta_1}, \frac{\psi_2(\theta)}{\Delta_2}, \dots \frac{\psi_{n-1}(\theta)}{\Delta_{n-1}}\right),$$

where

$$\psi_r(\theta) = \theta^r + a_{r1}\theta^{r-1} + \dots + a_{rr},$$

a number of *rank* r, and $\Delta_1, \Delta_2, \ldots \Delta_{n-1}$, are factors of the discriminant $D(\theta)$. Putting

$$\Delta_r = p^{\mu_r}\delta_r, \text{ with } dv(p, \delta_r) = 1,$$

and

$$a_{rs} \equiv c_{rs} \pmod{p^{\mu_r}},$$

it is first shewn that the prime factors of $D(\theta)$ can be separated, and that an integral basis can be immediately constructed when all complex integers

$$\frac{\Phi_r(\theta)}{p^{\mu_r}} \equiv \frac{\theta^r + c_{r1}\theta^{r-1} + \ldots + c_{rr}}{p^{\mu_r}}, \; r = 1, 2, \ldots n-1,$$

have been determined for each prime p dividing $D(\theta)$. Thus it is sufficient to confine the discussion to the evaluation of the integers

$$1, \; \frac{\Phi_1(\theta)}{p^{\mu_1}}, \; \frac{\Phi_2(\theta)}{p^{\mu_2}}, \; \ldots \; \frac{\Phi_r(\theta)}{p^{\mu_r}}, \; \ldots \; \frac{\Phi_{n-1}(\theta)}{p^{\mu_{n-1}}},$$

which form the *partial basis* of $\mathfrak{o}$ (bas p). Since $\theta\Phi_r(\theta)/p^{\mu_r}$ is integral

$$\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_r \leqslant \ldots \leqslant \mu_{n-1},$$

and, when $\mu_r = \mu_{r+1}$, it is enough to take $\Phi_{r+1}(\theta) = \theta\Phi_r(\theta)$. Inverting the notation, there is a minimum rank $r \equiv r_\mu$ which admits a complex integer

$$\frac{\phi_r(\theta)}{p^{\mu}} \equiv \frac{\theta^r + c_{r1}\theta^{r-1} + \ldots + c_{rr}}{p^{\mu}},$$

and, when

$$\mu_{r-1} < \mu_r = \mu_{r+1} = \ldots = \mu_{r+s},$$

it is sufficient to take

$$\Phi_{r+i}(\theta) = \theta^i\phi_r(\theta), \; i = 1, 2, \ldots s.$$

The $1 + l$ integers

$$1, \; \frac{\phi_1(\theta)}{p}, \; \frac{\phi_2(\theta)}{p^2}, \; \ldots \; \frac{\phi_l(\theta)}{p^l},$$

form the *stem* of the partial basis of $\mathfrak{o}$ (bas p) and in this investigation the partial basis is approached through its stem.

3. Expressing $a(z)$ as a product of irreducible divisors (mod p) in the form

$$a(z) \equiv \varpi_1(z)^{f_1}\varpi_2(z)^{f_2} \ldots \varpi_w(z)^{f_w} \pmod{p},$$

it is shewn that

$$\phi_\mu(z) \equiv \varpi_1(z)^{(\mu,1)}\varpi_2(z)^{(\mu,2)} \ldots \varpi_w(z)^{(\mu,w)} \pmod{p},$$

with

$$1 \leqslant (\mu, i) \leqslant (\mu+1, i) \leqslant f_i, \; i = 1, 2, \ldots w.$$

The polynomial $a(z)$ admits a unique expansion as a product of w

irreducible divisors $(\bmod\, p^{d+1})$ of which the typical factor, corresponding to the divisor $\varpi(z)^f\,(\bmod\, p)$, is

$$\left.\begin{array}{l} \varpi(z)^f \\ \qquad + p\,\zeta_{f-1\,1}(z)\,\varpi(z)^{f-1} + p\,\zeta_{f-2\,1}(z)\,\varpi(z)^{f-2} + \ldots + p\,\zeta_{01}(z) \\ \qquad + p^2\zeta_{f-1\,2}(z)\,\varpi(z)^{f-1} + p^2\,\zeta_{f-2\,2}(z)\,\varpi(z)^{f-2} + \ldots + p^2\zeta_{02}(z) \\ + \ldots + p^d\,\zeta_{f-1\,d}(z)\,\varpi(z)^{f-1} + p^d\,\zeta_{f-2\,d}(z)\,\varpi(z)^{f-2} + \ldots + p^d\,\zeta_{0\,d}(z) \end{array}\right\} \text{(A).}$$

Each polynomial $\zeta_{xy}(z)$ is of degree $g-1$ or lower, g being the degree of $\varpi(z)$, and, when the coefficients of $\zeta_{xy}(z)$ are all divisible by p, the term $p^y\zeta_{xy}(z)\,\varpi(z)^x$ is absorbed in $p^{y+1}\zeta_{x\,y+1}(z)\,\varpi(z)^x$.

Taking a positive quarter-plane cut up into unit squares, the *node* (x, y) is to be marked or unmarked according as $\zeta_{xy}(z)$ is different from or equal to zero. Since $a(z)$ is irreducible there is at least one marked node on the p-axis when d is great enough. A boundary of the diagram, consisting of a number of straight edges (which with the axes form a closed polygon) is next drawn, just as in Newton's method of separating the branches of a plane curve at a multiple point. With a modified suffix-notation the terms represented by nodes on a typical edge are

$$p^\sigma\varpi(z)^\rho\{\zeta_0(z)\,\varpi(z)^{ju} + p^v\zeta_1(z)\,\varpi(z)^{(j-1)u} + \ldots + p^{jv}\zeta_j(z)\} \equiv p^\sigma\varpi(z)^\rho Z(z),$$
$$dv(u, v) = 1,$$

the extreme coefficients $\zeta_0(z)$, $\zeta_j(z)$, and generally some others, being different from zero. On defining the modulus W^t to contain all integral polynomials

$$p^y\zeta_{xy}(z)\,\varpi(z)^x \text{ with } xv + yu \geqslant t,\ x > 0,\ y > 0,$$

it is shewn that $Z(z)$ is uniquely expressible as a product of irreducible divisors $(\bmod\, W^{juv+1})$, viz.:

$$Z(z) \equiv \zeta_0(z)\,\Xi(z)^M\,\Xi'(z)^{M'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

with

$$\Xi(z) = \varpi(z)^{mu} + p^v\xi_1(z)\,\varpi(z)^{(m-1)u} + \ldots + p^{mv}\xi_m(z),$$

and

$$Mm + M'm' + \ldots = j.$$

The presence of the divisor $\Xi(z)^M$ is shewn to involve the existence of an ideal $\mathfrak{j}$ of degree Mmg which divides p just u times and $\varpi(\theta)$ just v times. Further, mg is an aliquot part of the degree of every prime ideal factor of $\mathfrak{j}$. The prime ideals due to the factor $\Xi(z)^M$ are entirely distinct from those due to $\Xi'(z)^{M'}$, etc., so, in the particular case when *all* the indices M, M', etc. are unity, the prime ideal factors of p are completely separated.

If ν complete units on the line $y=\mu$ fall within the closed polygon, the terms in the first μ lines of (A) are all divisible by $\varpi(z)^\nu$ or $\varpi_i(z)^\nu$. Taking these terms to be

$$\varpi_i(z)^\nu \chi_{ik}(z),$$

it is shewn that

$$\chi_{1k}(\theta)\,\chi_{2k}(\theta)\cdots\chi_{wk}(\theta)/p^k,$$

is a complex integer. Further, when *all* the indices M, M', etc. are unity, it is proved that

$$\phi_k(\theta)=\chi_{1k}(\theta)\,\chi_{2k}(\theta)\cdots\chi_{wk}(\theta).$$

Most numerical cases, except those specially constructed, are covered by the assumption that *all* the indices M, M', etc. are unity, so, on this occasion, no further approximations are considered.

4. An ideal of $[\theta]$ is expressible in Dedekind's notation as a compound modulus of order n. From the expansion (A) bases of the prime ideal factors of p can also be set up immediately, assuming, as before, that M, M', etc. are all unity. To determine the prime ideal factors of a prime number q it is necessary to discover the irreducible divisors of $a(z)\,(\text{mod}\,q)$. These irreducible divisors (which also define the Galoisian imaginaries) have been used by Selling, Zolotareff and others in attempting to extend Kummer's theory of ideal numbers in cyclotomic fields to the general field of algebraic numbers. The object of these writers' work (to be first achieved by Dedekind through the introduction of the compound modulus) was to obtain a means of expressing every rational prime number as a unique product of (actual or ideal) prime factors in any field of algebraic numbers. Zolotareff's theory is only imperfect through its failure to account satisfactorily for the peculiarities of the prime factors of $D(\theta)$.

5. The reasoning of Chaps. VI–IX, by which the above results are obtained, is of a very general arithmetical character. It can be applied, for example, to the fields of algebraic functions of one variable discussed by Dedekind and Weber, a simplification then arising through the prime ideals being all of the first degree. An integral basis of a relative number-field can also be obtained by a process which follows the main argument except for obvious adjustments. The discussion of relative fields leads to results of which some have already been obtained by Wahlin.

A few numerical examples have been added to illustrate the power of the method when applied to a concrete case. Finally in Chap. XV an integral basis of $[\sqrt[n]{a}]$ is obtained for all values of a and n.

The outstanding question, when the indices M, M', etc. are not all equal to unity, bears no relation to the degree of the field. It presents much the same type of difficulty as that arising in the separation of those branches of a plane curve which have multiple contact with each other. But whereas general analytical theorems can be brought to bear on this latter question, an existence theorem has to be proved at every stage in the arithmetical problem considered here. Every failing case can be solved by a suitable rational transformation. Methods of discovering transformations applicable to an assigned numerical failing case have not yet been investigated.

CHAPTER II

DEDEKIND'S THEORY OF IDEAL NUMBERS

6. The theory of ideal numbers originated in early investigations on Fermat's last theorem. Lamé, in 1847, claimed to have proved that, if p is an odd prime and α a primitive p-th root of unity, the equation

$$x^p + y^p = z^p$$

is not satisfied when x, y, z are of the form

$$a_0 + a_1\alpha + \dots + a_{p-2}\alpha^{p-2},$$

the a's being rational integers. It was pointed out by Liouville, however, that Lamé had failed to prove that such a *complex integer* could be uniquely expressed as a product of prime factors of the same type, an omission which rendered the claim invalid. Kummer's researches into Fermat's last theorem led him to define *ideal numbers* in the cyclotomic field $[\alpha]$ given by

$$\alpha^{p-1} + \dots + \alpha + 1 = 0.$$

The definition led to a unique representation of any complex integer in the field as a product of prime ideal numbers, also to a proof of the impossibility of Fermat's equation in certain cases.

Kummer's system of ideals served his main purpose but was insufficient to deal with the algebraic numbers defined by any given algebraic equation. Dedekind first succeeded in constructing a definition of ideal numbers which renders an integer in any field of algebraic numbers uniquely expressible as a product of prime ideal factors. We sketch the main points in Dedekind's theory, so far as it is needed in the investigation which follows.

7. An *algebraic (complex) integer* of degree n is defined to be a root of an irreducible equation

$$a(\theta) \equiv \theta^n + a_1\theta^{n-1} + \ldots + a_n = 0,$$

whose coefficients a_r are rational integers. When the equation $a(\theta)=0$ is reducible each irreducible factor can be taken to have integral coefficients with unity as the first one (Gauss, *D. A.*, § 42). The roots of such a reducible equation are thus complex integers.

If α, β are algebraic integers of degrees n, m it is readily seen that

$$\Pi(x-\alpha-\beta), \quad \Pi(x-\alpha+\beta), \quad \Pi(x-\alpha\beta),$$

are polynomials of degree nm with rational integral coefficients, and hence that $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$ are integers—of degree nm or lower. More generally the number

$$\Sigma c_{ij}\alpha^i\beta^j, \quad i \geqslant 0, \; j \geqslant 0,$$

is integral provided i, j, c_{ij} are rational integers.

The field $[\theta]$ contains all numbers which can be derived from θ by any finite combination of the four elementary operations of arithmetic. Its general element is

$$\omega = (b_0\theta^r + b_1\theta^{r-1} + \ldots + b_r)/(g_0\theta^s + g_1\theta^{s-1} + \ldots + g_s),$$

where b_i, g_j are rational and the exponents positive integers: in virtue of $a(\theta)=0$ ω is expressible as

$$e_1\theta^{n-1} + e_2\theta^{n-2} + \ldots + e_n,$$

where the e's are rational. When θ is integral so are

$$\theta^2, \theta^3, \ldots, \theta^{n-1},$$

also

$$\omega_0 \equiv c_1\theta^{n-1} + c_2\theta^{n-2} + \ldots + c_n,$$

for rational integral values of the c's. In some circumstances, which it is the present purpose to investigate, ω_0 is also a complex integer for particular rational fractional values of the coefficients c_i.

8. Taking $\alpha_1, \alpha_2, \ldots, \alpha_m$ to be members of a field $[\theta]$, the aggregate of numbers of the type

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_m\alpha_m,$$

where $c_1, c_2, \ldots, c_m$ are rational integers, is called a compound modulus of order m and denoted by

$$\mathfrak{m} \equiv [\alpha_1, \alpha_2, \ldots \alpha_m].$$

If $|c_{ij}|$ is an integral unit determinant and

$$\beta_i = c_{1i}\alpha_1 + c_{2i}\alpha_2 + \ldots + c_{mi}\alpha_m, \quad i = 1, 2, \ldots m,$$

it is apparent that every member of $\mathfrak{m}$ belongs also to

$$\mathfrak{m}' \equiv [\beta_1, \beta_2, \ldots \beta_m]$$

and vice-versa; in fact $[\alpha_1, \alpha_2, \dots \alpha_m]$ and $[\beta_1, \beta_2, \dots \beta_m]$ are interchangeable bases of the same modulus.

The order of every compound modulus is n at most, for rational integers $c_{11}, c_{21}, \dots c_{n+1 1}$, with no common factor can be found to make

$$c_{11}\alpha_1 + c_{21}\alpha_2 + \dots + c_{n+1 1}\alpha_{n+1} = 0.$$

Choosing an integral unit determinant $|c_{ij}|$ of order $n+1$, put

$$\gamma_i = c_{1i}\alpha_1 + c_{2i}\alpha_2 + \dots + c_{n+1 i}\alpha_{n+1},$$

then

$$[\alpha_1, \alpha_2, \dots \alpha_{n+1}] = [\gamma_2, \dots \gamma_{n+1}],$$

which is of order n. It is evident that the aggregate of complex integers in the field is a compound modulus of order n. The basis of this modulus is

$$\mathfrak{o} \equiv (1, \omega_2, \dots \omega_n),$$

the integral basis of the field.

9. A compound modulus $[\alpha_1, \alpha_2, \dots, \alpha_n]$ is said to be an *ideal* when

(1) all its members are complex integers, so that the typical member is

$$\alpha_i = a_{1i} + a_{2i}\omega_2 + \dots + a_{ni}\omega_n,$$

for rational integral values of the a's, and

(2) $\omega\alpha_i$ belongs to the modulus for every complex integer ω of the field. The *norm* $N(\mathfrak{a})$ of the ideal

$$\mathfrak{a} \equiv (\alpha_1, \alpha_2, \dots, \alpha_n),$$

is the determinant $|a_{ji}|$ taken positively, and hence a positive rational integer.

An ideal $\mathfrak{a}$ is divisible by an ideal $\mathfrak{c}$ when every member of $\mathfrak{a}$ is also a member of $\mathfrak{c}$. A prime ideal is one which is only divisible by itself and $\mathfrak{o}$.

The product of two ideals

$$\mathfrak{a} \equiv (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \mathfrak{b} \equiv (\beta_1, \beta_2, \dots, \beta_n)$$

is the modulus $(\alpha_1\beta_1, \dots, \alpha_j\beta_i, \dots)$ which is expressible as a modulus with an n-member basis

$$\mathfrak{ab} = (\gamma_1, \gamma_2, \dots, \gamma_n), \text{ say.}$$

Subject to these definitions it was proved by Dedekind that

$$N(\mathfrak{ab}) = N(\mathfrak{a})\,N(\mathfrak{b}),$$

and that every ideal is uniquely expressible as a product of prime-ideal factors. Much of the pioneering difficulty in proving the latter of these theorems lay in constructing a correct definition of an ideal. If we assume, as the early investigators did, that the integral basis of $[\sqrt{17}]$ is $(1, \sqrt{17})$ then

$$(2, 1+\sqrt{17})^2 = (4, 2+2\sqrt{17}),$$

and the relation $N(\mathfrak{a})\, N(\mathfrak{b}) = N(\mathfrak{ab})$ breaks down. The attempts to prove uniqueness of factorisation made by Zolotareff, Laurent and others failed in this essential point.

To give an example of an actual system of ideals, take the quadratic field $[\sqrt{-23}]$ of which an integral basis is

$$\mathfrak{o} \equiv \left(1, \frac{1+\sqrt{-23}}{2}\right) \text{ or } (1, \alpha)$$

where $\alpha^2 - \alpha + 6 = 0$. The integer 6 can be expressed as a product of integral factors in two ways, viz.:

$$6 = 2 \,.\, 3 = \alpha(1-\alpha),$$

while none of the four factors $2, 3, \alpha, 1-\alpha$ is decomposable into integral factors. But the ideals $(2, 2\alpha)$ etc. are each expressible as a product of prime ideal factors, in fact

$$(2, 2\alpha) = (2, \alpha)(2, 1-\alpha),$$
$$(3, 3\alpha) = (3, \alpha)(3, 1-\alpha),$$
$$(\alpha, \alpha^2) = (6, \alpha) = (2, \alpha)(3, \alpha),$$
$$(1-\alpha, \alpha-\alpha^2) = (6, 1-\alpha) = (2, 1-\alpha)(3, 1-\alpha).$$

Thus

$$(6, 6\alpha) = (2, 2\alpha)(3, 3\alpha) = (6, \alpha)(6, 1-\alpha)$$

is the product of four prime ideals, and it can be similarly verified that any other ideal of $[\alpha]$ is uniquely expressible as a product of prime-ideal factors.

10. An ideal $(\beta, \beta\omega_2, \ldots, \beta\omega_n)$, which is the modulus of all complex integral multiples of β, is called a principal ideal and denoted by (β). Two ideals $\mathfrak{a}$, $\mathfrak{b}$ are said to be equivalent, and to belong to the same class, when an ideal $\mathfrak{c}$ exists so that $\mathfrak{ac}$, $\mathfrak{bc}$ are both principal.

Fundamental results proved by Dedekind are that the number of classes of ideals in any field of algebraic numbers is finite and that the prime factors of the *discriminant* (§ 12) are the only rational integers divisible by the square of a prime ideal.

A unit ϵ is a root of an equation

$$\epsilon^m + c_1\epsilon^{m-1} + c_2\epsilon^{m-2} + \ldots \pm 1 = 0,$$

with rational integral coefficients. If the irreducible equation defining the field has r real roots and $2s$ complex ones the totality of units is

$$\epsilon_1^{x_1}\epsilon_2^{x_2}\dots\epsilon_{r+s-1}^{x_{r+s-1}},$$

where the x's are rational integers and $\epsilon_1, \epsilon_2, \dots, \epsilon_{r+s-1}$ is a fundamental set of units, i.e. a set connected by no relation of the type

$$\epsilon_1^{z_1}\epsilon_2^{z_2}\dots\epsilon_{r+s-1}^{z_{r+s-1}}=1,$$

other than

$$z_1=z_2=\dots=z_{r+s-1}=0.$$

Every ideal is the highest common factor of two suitably chosen principal ideals (α) and (β). Hilbert, Hurwitz and some other writers use the notation $\mathfrak{j}=(\alpha, \beta)$, the ideal being the aggregate of complex integral multiples of α, β. Throughout the following pages we adhere to Dedekind's notation of the compound modulus with an n-member base.

CHAPTER III

CANONICAL FORM OF AN INTEGRAL BASIS

11. If θ is any one of the n conjugate roots $\theta_1, \theta_2, \dots \theta_n$ of the rational integral irreducible equation

$$a(\theta)\equiv\theta^n+a_1\theta^{n-1}+a_2\theta^{n-2}+\dots+a_n=0 \dots\dots\dots\dots\dots(1)$$

every member of the field $[\theta]$ is of the form

$$\omega\equiv b_1\theta^{n-1}+b_2\theta^{n-2}+\dots+b_n$$

with $b_1, b_2, \dots b_n$ rational. When ω is an algebraic integer it is necessary, but not sufficient, that

$$b_i=B_i/D(\theta)$$

where B_i is a rational integer and

$$D(\theta)=\Pi(\theta_r-\theta_s)^2. \quad (\textit{Zahlbericht}, 180.)$$

In a numerical case the discriminant $D(\theta)$ is calculated from the formula

$$D(\theta)=\begin{vmatrix} 1, & \theta_1, & \dots & \theta_1^{n-1} \\ 1, & \theta_2, & \dots & \theta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1, & \theta_n, & \dots & \theta_n^{n-1} \end{vmatrix}^2 = \begin{vmatrix} n, & S(\theta), & \dots & S(\theta^{n-1}) \\ S(\theta), & S(\theta^2), & \dots & S(\theta^n) \\ \dots & \dots & \dots & \dots \\ S(\theta^{n-1}), & S(\theta^n), & \dots & S(\theta^{2n-2}) \end{vmatrix}.$$

For convenience I put $\Delta(\theta)=|D(\theta)|$.

Among the integers of $[\theta]$, now of the type

$$\Omega = \frac{B_1\theta^{n-1} + B_2\theta^{n-2} + \dots + B_n}{\Delta(\theta)},$$

there is one, viz.: θ^s, and may be several, such that

$$B_i = 0,\ i < n - s;\ 0 < B_{n-s} \leqslant \Delta(\theta);\ 0 \leqslant B_i < \Delta(\theta),\ i > n - s.$$

Taking A_{s0} to be the least value of B_{n-s} satisfying these inequalities and

$$\omega_s \equiv (A_{s0}\,\theta^s + A_{s1}\,\theta^{s-1} + \dots + A_{ss})/\Delta(\theta),$$

a number of *rank* s, to be a corresponding complex integer, there are defined n integers

$$\omega_0,\ \omega_1,\ \omega_2,\ \dots\ \omega_{n-1}$$

of the respective ranks $0, 1, 2, \dots n-1$. If then

$$\Omega \equiv (B_1\theta^{n-1} + B_2\theta^{n-2} + \dots + B_n)/\Delta(\theta)$$

is any integer of the field, $B_1 \equiv 0 \pmod{A_{n-10}}$, otherwise suitable rational integers α, β would make

$$0 < \alpha A_{n-10} + \beta B_1 < A_{n-10}$$

and, $\alpha\omega_{n-1} + \beta\Omega$ being integral, A_{n-10} would not be the least value of B_1, contrary to supposition. Hence

$$B_1 = m_{n-1} A_{n-10}$$

and $$\Omega - m_{n-1}\,\omega_{n-1} = (B_2'\,\theta^{n-2} + \dots + B_n')/\Delta(\theta).$$

Repeating the argument it follows that

$$\Omega = m_{n-1}\,\omega_{n-1} + m_{n-2}\,\omega_{n-2} + \dots + m_0\,\omega_0 \qquad \text{..............(2)}$$

for rational integral values of $m_{n-1}, m_{n-2}, \dots m_0$. Moreover every number of the form (2) for rational integral values of m_i is an algebraic integer, and there can be no identical relation of the type

$$m_{n-1}\,\omega_{n-1} + m_{n-2}\,\omega_{n-2} + \dots + m_0\,\omega_0 = 0$$

unless every m_i is zero. Thus every integer of the field is uniquely expressible in the form (2), whence the integers of the field form a *compound modulus*

$$\mathbf{o} \equiv (\omega_0,\ \omega_1,\ \dots\ \omega_{n-1}) \qquad \text{........................(3)}$$

of order n; $\mathbf{o}$ is also an *integral basis* of the field.

12. Next we examine some properties of the coefficients A_{si}.

I. The only rational integers of $[\theta]$ being $0, \pm 1, \pm 2$, etc.,

$$A_{00} = \Delta(\theta) \text{ and } \omega_0 = 1.$$

II. Since $\theta^s \equiv \Delta(\theta)\,\theta^s/\Delta(\theta)$ is integral, $\Delta(\theta) \equiv 0 \pmod{A_{s0}}$. I put

$$\Delta(\theta) = A_{s0}\Delta_s.$$

III. Since $\theta\omega_{s-1}$ is integral, $A_{s-10} \equiv 0 \pmod{A_{s0}}$, whence

$$\Delta_s \equiv 0 \pmod{\Delta_{s-1}}.$$

IV. Each of the coefficients $A_{s1}, A_{s2}, \dots A_{ss}$ is divisible by A_{s0}. This result being true when $s=1$, for

$$\Delta_1\omega_1 - \theta \equiv A_{11}/A_{10}$$

is an integer I assume its truth for $s = 1, 2, \dots r-1$. Since $A_{r-10}\,\omega_r/A_{r0}$, $\theta\omega_{r-1}$ are integral,

$$A_{r-10}\,\omega_r/A_{r0} - \theta\omega_{r-1} = y_{r-1}\,\omega_{r-1} + \dots + y_0\,\omega_0$$

for rational integral values of $y_0, y_1, \dots y_{r-1}$,

i.e.
$$\frac{A_{r0}\theta^r + A_{r1}\theta^{r-1} + \dots + A_{rr}}{A_{r0}} = \theta\,\frac{A_{r-10}\,\theta^{r-1} + \dots + A_{r-1r-1}}{A_{r-10}} + \sum_{0,\,r-1} y_s\,\frac{A_{s0}}{A_{r-10}}\,\frac{A_{s0}\theta^s + \dots + A_{ss}}{A_{s0}}.$$

By supposition $A_{si} \equiv 0 \pmod{A_{s0}}$, and $A_{s0} \equiv 0 \pmod{A_{r-10}}$. Hence, on the left, $A_{ri} \equiv 0 \pmod{A_{r0}}$ and the result stated follows by induction.

Now putting

$$A_{si} = a_{si}\,A_{s0}$$

and
$$\psi_s(\theta) = \theta^s + a_{s1}\theta^{s-1} + \dots + a_{ss},$$

the integral basis $(\omega_0, \omega_1, \dots \omega_{n-1})$ takes the form

$$\left(1, \frac{\psi_1(\theta)}{\Delta_1}, \frac{\psi_2(\theta)}{\Delta_2}, \dots \frac{\psi_{n-1}(\theta)}{\Delta_{n-1}}\right) \dots\dots\dots\dots\dots\dots (4).$$

The discriminant of the field is

$$D \equiv D(\omega_0^{(i)}, \omega_1^{(j)}, \omega_2^{(k)}, \dots \omega_{n-1}^{(q)})$$
$$\equiv D(\theta)/\Delta_1^2\,\Delta_2^2 \dots \Delta_{n-1}^2.$$

CHAPTER IV

SEPARATION OF THE PRIME FACTORS OF $\Delta(\theta)$: THE PARTIAL BASIS OF o (Bas p) AND ITS STEM

13. Since each member of the sequence

$$\Delta_1, \Delta_2, \dots \Delta_{n-1}, \Delta(\theta)$$

is a factor of all that succeed it,

$$\Delta_r = p_1^{[1,\,r]}\,p_2^{[2,\,r]} \dots p_m^{[m,\,r]},$$
$$\Delta(\theta) = p_1^{\lambda_1}\,p_2^{\lambda_2} \dots p_m^{\lambda_m},$$

where $p_1, p_2, \dots p_m$ are different primes and

$$[i, 1] \leqslant [i, 2] \leqslant \dots \leqslant [i, n-1] < \lambda_i, \quad i = 1, 2, \dots m.$$

Further, since $\dfrac{\Delta_r}{p_1^{[1,\,r]}},\ \dfrac{\Delta_r}{p_2^{[2,\,r]}},\ \dots\ \dfrac{\Delta_r}{p_m^{[m,\,r]}}$ have no common factor, there exist rational integers $x_1, x_2, \dots x_m$ such that

$$\sum_{1,\,m} x_i \frac{\Delta_r}{p_i^{[i,\,r]}} = 1, \quad \text{whence} \quad \sum_{1,\,m} x_i \frac{\psi_r(\theta)}{p_i^{[i,\,r]}} = \frac{\psi_r(\theta)}{\Delta_r}.$$

If the coefficients of a new integral polynomial

$$\Phi_r^i(\theta) = \theta^r + c_{r1}^i\,\theta^{r-1} + \dots + c_{rr}^i$$

are such that

$$c_{rj}^i \equiv a_{rj} \pmod{p_i^{[i,\,r]}},$$

the number $\Phi_r^i(\theta)/p_i^{[i,\,r]}$ is integral. Also, when the numbers

$$\frac{\Phi_r^i(\theta)}{p_i^{[i,\,r]}} \qquad \begin{array}{l} i = 1, 2, \dots m, \\ r = 1, 2, \dots n-1, \end{array}$$

are known, the numbers $\psi_r(\theta)/\Delta_r$ are completely determined for the m congruences

$$a_{rj} \equiv c_{rj}^i \pmod{p_i^{[i,\,r]}}, \quad i = 1, 2, \dots m,$$

define a_{rj} uniquely (mod Δ_r).

The n complex integers

$$1,\ \frac{\Phi_1^i(\theta)}{p_i^{[i,\,1]}},\ \frac{\Phi_2^i(\theta)}{p_i^{[i,\,2]}},\ \dots\ \frac{\Phi_{n-1}^i(\theta)}{p_i^{[i,\,n-1]}},$$

will be said to form the *partial basis* of $\mathfrak{o}$ (bas p_i): their discriminant contains the prime factor p_i to the same power as D, i.e. to the power

$$\lambda_i - 2\,.\,([i, 1] + [i, 2] + \dots + [i, n-1]).$$

Dropping the suffix and putting $[i, r] = \mu_r$ the partial basis of $\mathfrak{o}$ (bas p) takes the form

$$1,\ \frac{\Phi_1(\theta)}{p^{\mu_1}},\ \frac{\Phi_2(\theta)}{p^{\mu_2}},\ \dots\ \frac{\Phi_{n-1}(\theta)}{p^{\mu_{n-1}}} \quad \dots\dots\dots\dots\dots\dots(5).$$

When

$$\mu_{r-1} < \mu_r = \mu_{r+1} = \dots = \mu_{s-1} < \mu_s$$

it is sufficient to take

$$\Phi_{r+i}(\theta) = \theta^i\,\Phi_r(\theta), \quad i = 1, 2, \dots s-r-1,$$

and the partial basis then takes the form

$$\begin{array}{l}
1,\ \theta,\ \theta^2,\ \dots\ \theta^{a-1}, \\[6pt]
\dfrac{\Phi_a(\theta)}{p^{\mu_a}},\ \dfrac{\theta\,\Phi_a(\theta)}{p^{\mu_a}},\ \dots\ \dfrac{\theta^{b-a-1}\,\Phi_a(\theta)}{p^{\mu_a}}, \\[10pt]
\dfrac{\Phi_b(\theta)}{p^{\mu_b}},\ \dfrac{\theta\,\Phi_b(\theta)}{p^{\mu_b}},\ \dots\ \dfrac{\theta^{c-b-1}\,\Phi_b(\theta)}{p^{\mu_b}}, \\[10pt]
\dfrac{\Phi_c(\theta)}{p^{\mu_c}},\ \text{etc.,}
\end{array}$$

in which any row may contain one term only.

14. I next define

$$\phi_\mu(z) = z^r + c_1 z^{r-1} + \dots + c_r$$

to be the integral polynomial of least rank r such that $\phi_\mu(\theta)/p^\mu$ is an algebraic integer. The

$$1 + l \equiv 1 + [i, n-1] \equiv 1 + \mu_{n-1} \text{ integers}$$

$$1, \frac{\phi_1(\theta)}{p}, \frac{\phi_2(\theta)}{p^2}, \dots \frac{\phi_l(\theta)}{p^l} \quad \text{.....................(6)}$$

are to be called the *stem* of the partial basis of $\mathfrak{o}$ (bas p). When, as often happens,

$$\mu_a = 1, \quad \mu_b = 2, \quad \mu_c = 3$$

the first four terms of the stem are

$$1, \frac{\Phi_a(\theta)}{p}, \frac{\Phi_b(\theta)}{p^2}, \frac{\Phi_c(\theta)}{p^3}.$$

The passage from the partial basis to the stem has a greater significance than the simplification in the suffix notation. In the partial basis (5) the exponent μ_r is a one-valued arithmetical function of the rank r: in the stem (6) the rank of $\phi_\mu(\theta)$ is a one-valued arithmetical function of the index μ.

The investigation of Chaps. VI—VIII approaches the partial basis (bas p) through its stem and the integral basis as a combination of the various partial bases.

CHAPTER V

THE EXPANSION OF A POLYNOMIAL $(\mathrm{mod}\, p^{d+1})$

15. Before discussing the partial basis of $\mathfrak{o}$ (bas p) and its stem it is necessary to obtain certain properties of the divisors of $a(z)$ (mod p) and (mod p^{d+1}). Dedekind has fully developed the theory of these divisors (mod p) and there are many earlier memoirs. H. J. S. Smith's Report (§ 67 etc.) and Tschebyscheff's *Theory of Congruences* (German translation by Schapira) may also be consulted on the divisors (mod p). To put down the stem of $\mathfrak{o}$ (bas p) the expansion which follows is needed for each prime p whose square divides $D(\theta)$.

It is known that the polynomial

$$a(z) \equiv z^n + a_1 z^{n-1} + \dots + a_n$$

is uniquely expressible as a product of irreducible divisors (mod p) in the form

$$a(z) \equiv \varpi_1(z)^{f_1}\varpi_2(z)^{f_2}\dots\varpi_w(z)^{f_w} \pmod{p} \text{...........(7)}$$

with $$\varpi_i(z) = z^{g_i} + c_{i1}z^{g_i-1} + \dots + c_{ig_i}$$

and $$f_1g_1 + f_2g_2 + \dots + f_wg_w = n.$$

The polynomial $a(z)$ cannot, however, be uniquely expressed as a product of irreducible divisors (mod p^d) with $d > 1$,

e.g. $$z^2 - 17 \equiv (z-1)(z+1) \equiv (z+3)(z-3) \pmod{2^3}.$$

A unique development of $a(z)$ (mod p^{d+1}), d being arbitrary, is fundamental in what follows and will now be obtained.

16. First put

$$a(z) = \varpi_1(z)^{f_1}\varpi_2(z)^{f_2}\dots\varpi_w(z)^{f_w} + pV_1(z)$$

and $$\xi_i(z) = \frac{\varpi_1(z)^{f_1}\varpi_2(z)^{f_2}\dots\varpi_w(z)^{f_w}}{\varpi_i(z)^{f_i}}, \quad i = 1, 2, \dots w.$$

The polynomials $\xi_1(z), \xi_2(z), \dots \xi_w(z)$ having no common divisor (mod p), it follows by a simple extension of the Euclidean greatest common measure process that there exists a further set of integral polynomials

$$\eta_1'(z), \eta_2'(z), \dots \eta_w'(z),$$

such that

$$\eta_1'(z)\xi_1(z) + \eta_2'(z)\xi_2(z) + \dots + \eta_w'(z)\xi_w(z) \equiv 1 \pmod{p}.$$

Putting also $$\eta_i''(z) = \eta_i'(z)V_1(z),$$

we have

$$\eta_1''(z)\xi_1(z) + \eta_2''(z)\xi_2(z) + \dots + \eta_w''(z)\xi_w(z) \equiv V_1(z) \pmod{p}.$$

Since the degree of $V_1(z)$ is less than n, which

$$= f_1g_1 + f_2g_2 + \dots + f_wg_w,$$

that of $\eta_1''(z)$ can only exceed $f_1g_1 - 1$ when another polynomial, $\eta_2''(z)$ say, is of higher degree than $f_2g_2 - 1$. Putting

$$\eta_1''(z) = \varpi_1(z)^{f_1}Q(z) + \eta_{11}(z),$$

$$\eta_2''(z) = -\varpi_2(z)^{f_2}Q(z) + \eta_{12}'(z),$$

where $\eta_{11}(z)$ is of degree $f_1g_1 - 1$ or lower, then

$$\eta_{11}(z)\xi_1(z) + \eta_{12}'(z)\xi_2(z) + \dots + \eta_{1w}'(z)\xi_w(z) \equiv V_1(z) \pmod{p}.$$

Repeating the process when necessary we have

$$\eta_{11}(z)\xi_1(z) + \eta_{12}(z)\xi_2(z) + \dots + \eta_{1w}(z)\xi_w(z) \equiv V_1(z) \pmod{p}$$

with $\eta_{1i}(z)$ of degree $f_i g_i - 1$ at most. These polynomials $\eta_{1i}(z)$ are unique $(\text{mod } p)$, for if also

$$\epsilon_1(z)\,\xi_1(z) + \epsilon_2(z)\,\xi_2(z) + \dots + \epsilon_w(z)\,\xi_w(z) \equiv V_1(z) \pmod p,$$

then

$$\Sigma\,\{\epsilon_i(z) - \eta_{1i}(z)\}\,\xi_i(z) \equiv 0 \pmod p,$$

implying that $\varpi_i(z)^{f_i}$ divides $\epsilon_i(z) - \eta_{1i}(z)$ $(\text{mod } p)$, which, since $\epsilon_i(z)$, $\eta_{1i}(z)$ are of degree $f_i g_i - 1$ or lower, involves $\epsilon_i(z) \equiv \eta_{1i}(z)$ $(\text{mod } p)$.

Next putting

$$a(z) = \prod_{i=1}^{i=w} [\varpi_i(z)^{f_i} + p\eta_{1i}(z)] + p^2 V_2(z),$$

the coefficients of $V_2(z)$ are integral and a second set of integral polynomials

$$\eta_{21}(z),\ \eta_{22}(z),\ \dots\ \eta_{2w}(z),$$

unique $(\text{mod } p)$, and of degrees $f_1 g_1 - 1, \dots f_w g_w - 1$ or lower, can be found to make

$$\eta_{21}(z)\,\xi_1(z) + \dots + \eta_{2w}(z)\,\xi_w(z) \equiv V_2(z) \pmod p.$$

It follows that

$$a(z) = \prod_{i=1}^{i=w} [\varpi_i(z)^{f_i} + p\eta_{1i}(z) + p^2\eta_{2i}(z)] + p^3 V_3(z)$$

and, repeating the process to d stages,

$$a(z) = \prod_{i=1}^{i=w} [\varpi_i(z)^{f_i} + p\eta_{1i}(z) + \dots + p^d\eta_{di}(z)] + p^{d+1} V_{d+1}(z) \quad (8).$$

When the polynomials $\eta_{ji}(z)$, $j < d$, have been determined, the coefficients of

$$\eta_{d1}(z),\ \eta_{d2}(z),\ \dots\ \eta_{dw}(z)$$

are n in number and uniquely determined $(\text{mod } p)$ by the solution of n linear congruences $(\text{mod } p)$. These coefficients can be made absolutely unique, if desirable, by being restricted to lie within the limits $0, p-1$ inclusive. Whether such a restriction is made or not the coefficients of

$$\varpi_i(z)^{f_i} + p\eta_{1i}(z) + \dots + p^d\eta_{di}(z)$$

are definite $(\text{mod } p^{d+1})$. Any multiples of p^{s+1} in $p^s\eta_{si}(z)$ can be passed on to the next term $p^{s+1}\eta_{s+1i}(z)$ if necessary, and any terms from

$$p^2\eta_{2i}(z) + \dots + p^d\eta_{di}(z),$$

or indeed the whole expression, can be absorbed in $p\eta_{1i}(z)$, unless $\eta_{1i}(z) = 0$. The essential features are the uniqueness of the polynomial

$$\varpi_i(z)^{f_i} + p\eta_{1i}(z) + \dots + p^d\eta_{di}(z) \pmod{p^{d+1}},$$

and the fact that the degree of its last d terms is less than $f_i g_i$.

17. A further development of the expansion is necessary. Dropping the suffix, the divisor in (8) due to the typical irreducible divisor $\varpi(z)$ of $a(z)$ (mod p) is

$$a'_{d+1}(z) \equiv \varpi(z)^f + p\eta_1(z) + \dots + p^d \eta_d(z) \pmod{p^{d+1}}.$$

The polynomial $\eta_1(z)$ is to be expanded in the form

$$\zeta'_{f-11}(z)\,\varpi(z)^{f-1} + \zeta'_{f-21}(z)\,\varpi(z)^{f-2} + \dots + \zeta'_{01}(z)$$

where $\zeta'_{i1}(z)$ is an integral polynomial of degree $g-1$ or lower. Reducing the coefficients of $\zeta'_{i1}(z)$ to integers less than p and not negative and absorbing any rejected terms in $p^2\eta_2(z)$, we have

$$\begin{aligned} a'_{d+1}(z) = \varpi(z)^f & \\ & + p\,[\zeta_{f-11}(z)\,\varpi(z)^{f-1} + \zeta_{f-21}(z)\,\varpi(z)^{f-2} + \dots + \zeta_{01}(z)] \\ & + p^2\eta_2'(z) + p^3\eta_3(z) + \dots + p^d\eta_d(z). \end{aligned}$$

The polynomial $\eta_2'(z)$ is to be similarly treated and the process continued to d stages. Finally

$$\begin{aligned} a_{d+1}(z) = a^{(i)}_{d+1}(z) = \varpi(z)^f & \\ & + p\,[\zeta_{f-11}(z)\,\varpi(z)^{f-1} + \zeta_{f-21}(z)\,\varpi(z)^{f-2} + \dots + \zeta_{01}(z)] \\ & + p^2\,[\zeta_{f-12}(z)\,\varpi(z)^{f-1} + \zeta_{f-22}(z)\,\varpi(z)^{f-2} + \dots + \zeta_{02}(z)] \\ + \dots & + p^d\,[\zeta_{f-1d}(z)\,\varpi(z)^{f-1} + \zeta_{f-2d}(z)\,\varpi(z)^{f-2} + \dots + \zeta_{0d}(z)] \\ \equiv a'_{d+1}(z) & \pmod{p^{d+1}} \quad \dots\dots\dots\dots (9), \end{aligned}$$

where each $\zeta_{ij}(z)$ is an integral polynomial of degree $g-1$ or lower, its coefficients lying between the limits $0, p-1$ inclusive.

Rearrangement of the terms in $\zeta_{ij}(z)$ violating these limitations on the coefficients is permissive under certain restrictions and sometimes desirable (*v.* Chap. XIV below).

Some of the polynomials $\zeta_{ij}(z)$ may vanish but, when $a(z)$ is irreducible, the set $\zeta_{01}(z), \zeta_{02}(z), \dots$ cannot all vanish. We suppose that

$$\zeta_{01}(z) = \zeta_{02}(z) = \dots = \zeta_{0h-1}(z) = 0,\ \zeta_{0h}(z) \neq 0 \dots\dots\dots (10).$$

When $\varpi(z)$ is replaced by $\varpi'(z)$, which $\equiv \varpi(z)$ (mod p), h is generally altered, a fact used hereafter (§ 75).

Due to the different irreducible divisors

$$\varpi_1(z),\ \varpi_2(z),\ \dots\ \varpi_w(z),$$

the respective values of h are $h_1, h_2, \dots h_w$ and H is the greatest of these. To put down the partial basis of $\mathfrak{o}$ (bas p) it is found necessary to carry the expansion (9) as far as (mod p^{H+1}).

CHAPTER VI

FIRST DISSECTION

THE PRIME IDEAL FACTORS OF (p)

18. We proceed to discuss some properties of the stem

$$1, \frac{\phi_1(\theta)}{p}, \frac{\phi_2(\theta)}{p^2}, \dots \frac{\phi_l(\theta)}{p^l}$$

of $\mathfrak{o}$ (bas p). It is convenient to consider the number

$$0 \equiv \frac{\phi_{l+1}(\theta)}{p^{l+1}} \equiv \frac{a(\theta)}{p^{l+1}},$$

of rank n, as the last term of the stem.

19. If $$\frac{c_0\theta^r + c_1\theta^{r+1} + \dots + c_r}{p^\mu},$$

a number of rank $r < r_\mu$, is integral then $c_0, c_1, \dots c_r$ are all divisible by p. This follows at once from the definition of the stem, for r_μ is the least rank which admits a complex integer

$$(\theta^s + c_1'\theta^{s-1} + \dots + c_s')/p^\mu.$$

20. Put $$a(z) = Q(z)\,\phi_\mu(z) + R(z),$$

with $R(z)$ of lower degree than $\phi_\mu(z)$, then

$$0 = \frac{a(\theta)}{p^\mu} = Q(\theta) \cdot \frac{\phi_\mu(\theta)}{p^\mu} + \frac{R(\theta)}{p^\mu}.$$

Since $Q(\theta) \cdot \phi_\mu(\theta)/p^\mu$ is integral, $R(\theta)/p^\mu$ is integral too and, (§ 19), every coefficient of $R(\theta)$ is divisible by p. Hence $\phi_\mu(z)$ divides $a(z)$ (mod p) and

$$\phi_\mu(z) \equiv \varpi_1(z)^{(\mu,1)}\,\varpi_2(z)^{(\mu,2)} \dots \varpi_w(z)^{(\mu,w)} \pmod{p}$$

with $$0 \leqslant (\mu, i) \leqslant f_i, \quad i = 1, 2, \dots w.$$

21. Put $$\phi_{\mu+1}(z) = Q_\mu(z)\,\phi_\mu(z) + R_\mu(z)$$

with $R_\mu(z)$ of degree $< r_\mu$, then

$$p \cdot \frac{\phi_{\mu+1}(\theta)}{p^{\mu+1}} - Q_\mu(\theta)\frac{\phi_\mu(\theta)}{p^\mu} = \frac{R_\mu(\theta)}{p^\mu}$$

and, the left side being integral, every coefficient of $R_\mu(\theta)$ is divisible by p. Hence $\phi_\mu(z)$ divides $\phi_{\mu+1}(z)$ (mod p) and

$$0 \leqslant (1, i) \leqslant \dots \leqslant (\mu, i) \leqslant (\mu+1, i) \leqslant \dots \leqslant f_i.$$

22. If $\phi_1(\theta)$ satisfies the irreducible equation

$$W(\theta) \equiv \phi_1(\theta)^e + a_1' \phi_1(\theta)^{e-1} + \dots + a_e' = 0$$

the coefficients $a_1', \dots a_e'$ are respectively divisible by $p, \dots p^e$. Since $a(z)$ is a factor of $W(z)$, $a(z)$ must divide $\phi_1(z)^e$ (mod p). Hence every irreducible divisor of $a(z)$ (mod p) divides $\phi_1(z)$ at least once, and the inequalities in § 21 take the more restricted form

$$1 \leqslant (1, i) \leqslant \dots \leqslant (\mu, i) \leqslant (\mu+1, i) \leqslant \dots \leqslant (l, i) \leqslant f_i \quad \dots\dots(11)$$

for $i = 1, 2, \dots w$.

23. Since $\dfrac{\phi_1(\theta)}{p} \cdot \dfrac{\phi_\mu(\theta)}{p^\mu}$ is a complex integer,

$$\phi_{\mu+1}(z) \text{ divides } \phi_1(z)\,\phi_\mu(z) \pmod p$$

and $$(1, i) + (\mu, i) \geqslant (\mu+1, i).$$

Similarly $$(\mu, i) + (\nu, i) \geqslant (\mu+\nu, i).$$

24. The polynomial $\phi_\mu(z)$ can be expanded (mod p^μ) by the method explained in Chap. v. Putting

$$j = (\mu, i)$$

the expansion is

$$\phi_\mu(z) = \prod_{i=1}^{i=w} \chi_\mu^{(i)}(z) + p^\mu \nu_\mu(z)$$

where $$\chi_\mu^{(i)}(z) = \varpi_i(z)^j$$

$$+\,p\,[\zeta^{(i)}{}_{j-11}(z)\,\varpi_i(z)^{j-1} + \zeta^{(i)}{}_{j-21}(z)\,\varpi_i(z)^{j-2} + \dots + \zeta^{(i)}{}_{01}(z)]$$
$$+\,p^2\,[\zeta^{(i)}{}_{j+12}(z)\,\varpi_i(z)^{j-1} + \zeta^{(i)}{}_{j-22}(z)\,\varpi_i(z)^{j-2} + \dots + \zeta^{(i)}{}_{02}(z)]$$
$$+ \dots + p^{\mu-1}[\zeta^{(i)}{}_{j-1\,\mu-1}(z)\,\varpi_i(z)^{j-1} + \zeta^{(i)}{}_{j-2\,\mu-1}(z)\,\varpi_i(z)^{j-2} + \dots + \zeta^{(i)}{}_{0\,\mu-1}(z)].$$

Unless $\mu = l+1$ no essential feature of $\phi_\mu(z)$ is lost by removing the term $p^\mu \nu_\mu(z)$. Omitting this term the standard expression for $\phi_\mu(z)$ is given by

$$\phi_\mu(z) = \chi_\mu^{(1)}(z)\,\chi_\mu^{(2)}(z) \dots \chi_\mu^{(w)}(z).$$

In particular,

$$\phi_1(z) = \varpi_1(z)^{(1,1)}\,\varpi_2(z)^{(1,2)} \dots \varpi_w(z)^{(1,w)}.$$

25. The number

$$\frac{\phi_1(\theta)}{p} \equiv \frac{\varpi_1(\theta)^{(1,1)}\,\varpi_2(\theta)^{(1,2)} \dots \varpi_w(\theta)^{(1,w)}}{p}$$

being integral, each prime ideal factor of (p) must divide $\phi_1(\theta)$ at least as often as it divides p. Hence every prime ideal factor of p divides one of the integers

$$\varpi_1(\theta),\ \varpi_2(\theta), \dots \varpi_w(\theta)$$

once at least. Further, each of these w integers contains at least one prime ideal factor of p; for, were $\varpi_1(\theta)$ free from such factors, the number

$$\varpi_2(\theta)^{(1,\,2)} \dots \varpi_w(\theta)^{(1,\,w)}/p$$

would be an integer, contrary to what has been proved in § 22.

26. If $\mathfrak{p}$, a prime ideal factor of $(p,\ \varpi_i(\theta))$, divides

$$\chi(\theta) \equiv c_0\theta^r + c_1\theta^{r-1} + \dots + c_r$$

then $\varpi_i(z)$ divides $\chi(z)$ (mod p). Otherwise, $\varpi_i(z)$ being irreducible (mod p), two integral polynomials $b(z)$, $c(z)$ could be found to make

$$b(z)\,\chi(z) + c(z)\,\varpi_i(z) = 1 + \lambda p,$$

and $\mathfrak{p}$, being a factor of $\chi(\theta)$, $\varpi_i(\theta)$, p, would divide unity.

It follows that no two of the integers

$$\varpi_1(\theta),\ \ \varpi_2(\theta), \dots \varpi_w(\theta)$$

are divisible by the same prime ideal factor of p.

27. If $\mathfrak{p}_1$, $\mathfrak{p}_2$, etc. are different prime ideals, some conciseness of expression will be gained by saying that an integer ω is *exactly divisible* by $\mathfrak{p}_1^x\,\mathfrak{p}_2^y \dots$ when ω is divisible by $\mathfrak{p}_1^x$ but not by $\mathfrak{p}_1^{x+1}$, by $\mathfrak{p}_2^y$ but not by $\mathfrak{p}_2^{y+1}$ and so on.

The prime ideal factors of $(p,\ \varpi_i(\theta))$ are $\mathfrak{p}_1$, $\mathfrak{p}_2$, ... $\mathfrak{p}_\tau$ and p is exactly divisible by

$$\mathfrak{p}_1^{u_1}\,\mathfrak{p}_2^{u_2} \dots \mathfrak{p}_\tau^{u_\tau},$$

$$\varpi_i(\theta) \text{ by } \mathfrak{p}_1^{v_1}\,\mathfrak{p}_2^{v_2} \dots \mathfrak{p}_\tau^{v_\tau}.$$

Since $\mathfrak{p}_j$ divides $\varpi_i(\theta)^{(1,\,i)}$ at least as often as it divides p,

$$u_j \leqslant v_j(1,\ i),\ \ j = 1,\ 2, \dots w.$$

Also $u_j > v_j\,[(1,\ i) - 1]$ for at least one j, otherwise each of the prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, ... $\mathfrak{p}_\tau$ would divide $\varpi_i(\theta)^{(1,\,i)-1}$ at least as often as p and

$$\varpi_1(\theta)^{(1,\,1)} \dots \varpi_i(\theta)^{(1,\,i)-1} \dots \varpi_w(\theta)^{(1,\,w)}/p$$

would be a complex integer.

The exact values of the ratios of $v_j : u_j$ are found in Chap. VII.

28. The number $\chi_\mu^{(i)}(\theta)$ is the integral polynomial in θ of lowest rank (with unit leading coefficient) which is divisible by each prime ideal factor of $(p,\ \varpi_i(\theta))$ at least as often as p^μ. For were $\chi'(\theta)$, a number of lower rank than $\chi_\mu^{(i)}(\theta)$, to contain each of these prime ideals as often as p^μ, the number

$$\frac{\chi'(\theta)}{\chi_\mu^{(i)}(\theta)} \cdot \frac{\phi_\mu(\theta)}{p^\mu},$$

of lower rank than $\phi_\mu(\theta)$, would be integral—contrary to the definition of $\phi_\mu(\theta)$.

29. It is now convenient to simplify the notation by putting

$$\varpi_i(z) = \varpi(z), \quad \varpi_i(\theta) = \varpi(\theta), \quad g_i = g, \quad f_i = f.$$

Taking $\mathfrak{p}_j$ to be one of the prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots \mathfrak{p}_r$, the congruence

$$x_1\theta^{g-1} + \dots + x_g \equiv 0 \pmod{\mathfrak{p}_j}$$

is only satisfied, in rational integers x_i, by

$$x_1 \equiv \dots \equiv x_g \equiv 0 \pmod{p}.$$

Should every complex integer ω of $[\theta]$ be congruent to

$$x_1\theta^{g-1} + \dots + x_g \pmod{\mathfrak{p}_j}$$

for suitable values of x_i the degree of $\mathfrak{p}_j$ is g and $N(\mathfrak{p}_j) = p^g$.

But when there exists a complex integer

$$\omega_1 \text{ which } \not\equiv x_1\theta^{g-1} + \dots + x_g \pmod{\mathfrak{p}_j}$$

for any rational integral values of x_i, the only solution in rational integers of

$$x_1\theta^{g-1} + \dots + x_g + (y_1\theta^{g-1} + \dots + y_g)\,\omega_1 \equiv 0 \pmod{\mathfrak{p}_j}$$

is
$$x_1 \equiv \dots \equiv x_g \equiv y_1 \equiv \dots \equiv y_g \equiv 0 \pmod{p}.$$

Otherwise suitable integral polynomials $b(z)$, $c(z)$ would make

$$b(z)(y_1z^{g-1} + \dots + y_g) + c(z)\,\varpi(z) \equiv 1 \pmod{p},$$

whence would

$$\omega_1 \equiv -b(\theta)(x_1\theta^{g-1} + \dots + x_g) \pmod{\mathfrak{p}_j},$$

an impossible congruence by definition of ω_1. Thus the degree of $\mathfrak{p}_j$ is $2g$ at least and, when every complex integer of the field is congruent to

$$x_1\theta^{g-1} + \dots + x_g + (y_1\theta^{g-1} + \dots + y_g)\,\omega_1 \pmod{\mathfrak{p}_j}$$

for suitable integral values of x_i, y_j, it is $2g$ exactly.

When the last condition is not satisfied, a repetition of the argument shews g must be an aliquot part of the degree d_j of $\mathfrak{p}_j$.

30. Since $\varpi(z)$ is irreducible (mod p) the equation

$$\varpi(\alpha) = 0$$

is irreducible and, if $\alpha_1, \dots \alpha_g$ are its roots, (p) is a prime ideal in each of the conjugate fields $[\alpha_i]$.

Now assuming, as in (10), that

$$\zeta_{0\,1}(z)=\zeta_{0\,2}(z)=\ldots=\zeta_{0\,h-1}(z)=0,\quad \zeta_{0\,h}(z)\neq 0,$$

the norm

$$\begin{aligned}&(-)^{ng}N[\varpi(\theta)]\\ &=(-)^{ng}N(\theta-a_1)\,N(\theta-a_2)\ldots N(\theta-a_g)\\ &=a(a_1)\,a(a_2)\ldots a(a_g)\\ &=\prod_{i=1}^{i=g}[p^h\zeta_{0\,h}(a_i)\,Q(a_i)+p^{h+1}R(a_i)]\\ &=p^{gh}\prod_{i=1}^{i=g}[\zeta_{0\,h}(a_i)\,Q(a_i)+pR(a_i)],\end{aligned}$$

where $Q(z)=\varpi_1(z)^{f_1}\ldots\varpi_{i-1}(z)^{f_{i-1}}\varpi_{i+1}(z)^{f_{i+1}}\ldots\varpi_w(z)^{f_w}$. Since neither $\zeta_{0\,h}(a_i)$ nor $Q(a_i)$ is divisible by the prime ideal (p) in $[a_i]$ the last expression is divisible by p^{gh} exactly. It follows that the degrees

$$d_1,\ d_2,\ \ldots\ d_\tau$$

of

$$p_1,\ p_2,\ \ldots\ p_\tau$$

are connected by the relation

$$d_1v_1+d_2v_2+\ldots+d_\tau v_\tau=gh \quad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

31. If q is a prime number, c a factor of $q-1$, and b a primitive root of the congruence

$$z^{q-1}-1\equiv 0\ (\mathrm{mod}\ q)$$

the polynomial z^c-b is irreducible (mod q). Taking q to be a prime of the linear form

$$Mfg+1,$$

but distinct from p, and adding to certain coefficients of

$$\begin{aligned}a_{h+1}(z)\equiv a^{(i)}{}_{h+1}(z)\equiv\ &\varpi(z)^f\\ &+p\ [\zeta_{f-1\,1}(z)\,\varpi(z)^{f-1}+\zeta_{f-2\,1}(z)\,\varpi(z)^{f-2}+\ldots+\zeta_{0\,1}(z)]\\ &+p^2[\zeta_{f-1\,2}(z)\,\varpi(z)^{f-1}+\zeta_{f-2\,2}(z)\,\varpi(z)^{f-2}+\ldots+\zeta_{0\,2}(z)]\quad\ldots(13)\\ +\ldots&+p^h[\zeta_{f-1\,h}(z)\,\varpi(z)^{f-1}+\zeta_{f-2\,h}(z)\,\varpi(z)^{f-2}+\ldots+\zeta_{0\,h}(z)]\end{aligned}$$

suitable multiples of p^{h+1} a new polynomial

$$X(z)+p^h\zeta_{0\,h}(z)$$

with

$$X(z)\equiv z^{fg}-b\ (\mathrm{mod}\ q)$$

is defined, and the equation $X(z)=0$ is irreducible. Each prime ideal factor of $(p,\ \varpi(\theta))$ divides

$$X(\theta)+p^h\zeta_{0\,h}(\theta)$$

at least as often as p^{h+1}: it therefore divides $X(\theta)$ just as often as p^h.

Taking $\beta_1, \beta_2, \dots \beta_{fg}$ to be the roots of $X(z)=0$, the norm $N[X(\theta)]$

$$\begin{aligned} &= N(\theta-\beta_1)\, N(\theta-\beta_2) \dots N(\theta-\beta_{fg}) \\ &= (-)^{fgn}\, a(\beta_1)\, a(\beta_2) \dots a(\beta_{fg}) \\ &= (-)^{fgn}\, p^{fgh} \prod_{i=1}^{i=fg} [-\zeta_{0\,h}(\beta_i)\, Q(\beta_i) + p\, R_1(\beta_i)], \end{aligned}$$

where $Q(z)$ is defined as in § 30. Since

$$X(z) \equiv \varpi(z)^f \pmod{p}$$

each prime ideal factor of p in $[\beta_i]$ divides $\varpi[\beta_i]$. The integer

$$-\zeta_{0\,h}(\beta_i)\, Q(\beta_i) + pR_1(\beta_i)$$

therefore contains no prime ideal factor of p in $[\beta_i]$, whence

$$N[X(\theta)] \equiv 0 \pmod{p^{fgh}} \text{ exactly.}$$

It follows that the total degree of the prime ideal factors of p dividing

$$X(\theta) \text{ or } p^h\, \zeta_{0\,h}(\theta) \text{ or } p^h \text{ is } fgh,$$

i.e.

$$h(d_1u_1 + d_2u_2 + \dots + d_\tau u_\tau) = fgh$$

or

$$d_1u_1 + d_2u_2 + \dots + d_\tau u_\tau = fg \qquad\qquad (14).$$

32. When p is prime to $D(\theta)$ there is no complex integer $\phi(\theta)/p$, neither is (p) divisible by the square of a prime ideal. The partial basis of $\mathfrak{o}$ (bas p) is

$$1,\ \theta,\ \theta^2,\ \dots\ \theta^{n-1}.$$

Further, the irreducible divisors of $a(z)$ (mod p) are all distinct, but the arguments of §§ 22, 25, 29, applied to the number

$$\varpi_1(\theta)\, \varpi_2(\theta) \dots \varpi_w(\theta)/p$$

of rank n, shew that

$$\varpi_1(\theta),\quad \varpi_2(\theta),\ \dots\ \varpi_w(\theta)$$

are each divisible by a prime ideal factor of (p), and that the degree of each prime ideal factor of $(p, \varpi_i(\theta))$ is a multiple of g_i. Since

$$g_1 + g_2 + \dots + g_w = n$$

it follows that

$$(p) = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_w,$$

$\mathfrak{p}_i$ being of degree g_i. Each member of the modulus

$$\mathfrak{m} \equiv (p, p\theta, \dots p\theta^{g_i-1},\ \varpi_i(\theta),\ \theta\,\varpi_i(\theta), \dots\ \theta^{n-1-g_i}\,\varpi_i(\theta))$$

is divisible by $\mathfrak{p}_i$. This modulus $\mathfrak{m}$ will be called the partial basis of $\mathfrak{p}_i$ (bas p) and the integers

$$p,\ \varpi_i(\theta)$$

form the stem of the partial basis of $\mathfrak{p}_i$ (bas p).

The first dissection therefore completely separates the prime ideal factors of a prime number p not contained in $D(\theta)$. It only supplies partial information about the integers $\phi(\theta)/p^\mu$ when $D(\theta) \equiv 0 \pmod{p^2}$ and only partially separates the prime ideal factors of such primes p.

CHAPTER VII

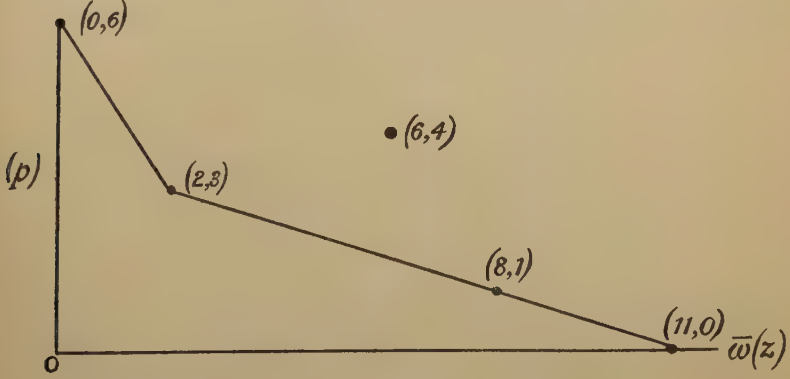
SECOND DISSECTION

THE PRIME IDEAL FACTORS OF $(p, \varpi(\theta))$

33. The positive quadrant of a Cartesian plane is cut up into unit squares by lines parallel to the axes and intersecting in points with positive integral co-ordinates, called *nodes.* When the expansion (13) contains a term $p^y \zeta_{xy}(z) \varpi(z)^x$, with $\zeta_{xy}(z) \neq 0$, the node (x, y) in the plane is to be marked. The same node is left unmarked when the corresponding term $\zeta_{xy}(z)$ is missing from the expansion. On the axis of abscissas the only marked node is $(f, 0)$ and of the marked nodes on the axis of ordinates $(0, h)$ is nearest the origin.

A straight edge, initially coinciding with the ϖ-axis, is to be pivoted at Q_0, $(f, 0)$, and turned in clockwise direction until it first crosses another marked node Q'. Q_0 is to be joined by a straight line to Q_1, the marked node on Q_0Q' which is most distant from Q_0. (If Q_0, Q' are the only marked nodes on Q_0Q', $Q' = Q_1$.) The straight edge is then pivoted at Q_1 and turned in the same direction as before until it again crosses a marked node Q''. Q_1 is joined by a straight line to Q_2, the marked node on Q_1Q'' which is furthest from Q_1. Repeating the process until the node Q_L, $(0, h)$, is reached, a broken line $Q_0Q_1Q_2 \dots Q_L$, of h or fewer segments, connects the two nodes $(f, 0)$, $(0, h)$, and the closed polygon $OQ_0Q_1Q_2 \dots Q_LO$ has no marked node inside it. The broken line $Q_0Q_1Q_2 \dots Q_L$ will be called the *boundary*, and the separate straight segments

$$E_1 = Q_0Q_1, \quad E_2 = Q_1Q_2, \dots E_L = Q_{L-1}Q_L$$



the *edges* of the boundary. If E_r joins the nodes $(x'+u', y')$, $(x', y'+v')$ the ratio v'/u' will be called the *slant* of E_r. In the figure, which corresponds to the expansion

$$a_7(z) \equiv \varpi(z)^{11} + p\zeta_{81}(z)\,\varpi(z)^8 + p^3\zeta_{23}(z)\,\varpi(z)^2$$
$$+ p^4\zeta_{64}(z)\,\varpi(z)^6 + p^6\zeta_{06}(z) \pmod{p^7},$$

there are two edges, of slants $\frac{1}{3}$ and $\frac{3}{2}$, connecting (11, 0) to (2, 3) and (2, 3) to (0, 6).

34. If $\mathfrak{p}$ is a prime ideal factor of $(p, \varpi(\theta))$ such that

$$p \equiv 0 \pmod{\mathfrak{p}^u} \text{ exactly}, \quad \varpi(\theta) \equiv 0 \pmod{\mathfrak{p}^v} \text{ exactly},$$

then
$$T \equiv p^y \zeta_{xy}(\theta)\,\varpi(\theta)^x \equiv 0 \pmod{\mathfrak{p}^{uy+vx}} \text{ exactly}.$$

Among all the terms in (13), after putting $z = \theta$, there must be two at least which are divisible by $\mathfrak{p}$ equally often but less often than any other term. Otherwise some one term

$$p^y \zeta_{xy}(\theta)\,\varpi(\theta)^x \text{ would} \equiv 0 \pmod{\mathfrak{p}^{uy+vx+1}}$$

which is impossible. If $T_1, T_2, \ldots$ are the terms divisible by $\mathfrak{p}$ equally often but less often than any other term T',

$$uy_1 + vx_1 = uy_2 + vx_2 = \ldots < uy' + vx'.$$

Hence the marked nodes (x_1, y_1), (x_2, y_2), ... lie on a line and every other marked node (x', y') lies on the opposite side of this line to the origin. It follows that this line is an edge of the boundary and that v/u is the slant of the edge. The edge and the prime ideal will be said to *attach* each other.

It has been proved already (§ 29) that g, the rank of $\varpi(\theta)$, is an aliquot part of the degree of every prime ideal factor of $(p, \varpi(\theta))$. Further, in §§ 30, 31, it has been shewn that the degrees of all these prime ideals and the indices of the exact powers to which they respectively divide p and $\varpi(\theta)$ are connected by the relations

$$u_1\frac{d_1}{g} + u_2\frac{d_2}{g} + \ldots\ldots\ldots + u_r\frac{d_r}{g} = f \quad \ldots\ldots\ldots\ldots(15),$$

$$v_1\frac{d_1}{g} + v_2\frac{d_2}{g} + \ldots\ldots\ldots + v_r\frac{d_r}{g} = h \quad \ldots\ldots\ldots\ldots(16).$$

Now it will be shewn that an edge whose end-nodes are $(x'+u', y')$, $(x', y'+v')$ must attach a set of prime ideals whose contributions to the sums on the left of (15) and (16) are u' and v' respectively.

35. The slant of the typical edge E_r is v_r/u_r where u_r, v_r are positive integers without common factor. For convenience in printing the suffix

r in u_r, v_r will be dropped up to the end of § 38. With an obvious modification in the suffix-notation those terms in (13) which are represented by nodes on E_r are

$$p^{\sigma} \varpi(z)^{\rho} [\zeta_0(z) \varpi(z)^{ju} + p^{v} \zeta_1'(z) \varpi(z)^{(j-1)u} + \ldots + p^{jv} \zeta_j'(z)].$$

If the edge includes unmarked nodes between its ends some of the terms $\zeta_1'(z), \ldots \zeta_{j-1}'(z)$ will be missing, but $\zeta_0(z)$, $\zeta_j'(z)$ are actually present in the expansion.

The equation of the edge E_r is

$$vx + uy = v\rho + u\sigma + juv.$$

Since u, v are co-prime the line l_t given by

$$vx + uy = v\rho + u\sigma + juv + t,$$

for integral values of t, also passes through nodes, and there is no node in the strip of plane between l_0 and l_1 or in that between l_t and l_{t+1}.

Putting

$$\zeta_i'(z) = \zeta_0(z) \zeta_i(z) + \varpi(z) W_i(z) + p W_i'(z), \quad i = 1, 2, \ldots j,$$

where $\zeta_i(z)$, $W_i(z)$, $W_i'(z)$ are of degree $g-1$ or lower, the terms in

$$p^{\sigma} \varpi(z)^{\rho} . \, p^{iv} \varpi(z)^{(j-i)u} [\varpi(z) W_i(z) + p W_i'(z)]$$

correspond to nodes on l_v, l_u and can be absorbed in the expansion without disturbing the edge E_r. With

$$Z_r(z) = \varpi(z)^{ju} + p^{v} \zeta_1(z) \varpi(z)^{(j-1)} + \ldots + p^{jv} \zeta_j(z)$$

there ensues the modified expansion

$$\left. \begin{array}{l} a_{h+1}(z) = p^{\sigma} \varpi(z)^{\rho} \zeta_0(z) Z_r(z) \\ \quad + \text{terms giving nodes on } l_1, l_2, \text{etc.} \end{array} \right\} \ldots\ldots\ldots\ldots\ldots (17).$$

36. The terms in (17) represented by nodes on l_1 are

$$\Sigma p^{y} \zeta'_{xy}(z) \varpi(z)^{x}$$

with $$vx + uy = v\rho + u\sigma + juv + 1.$$

Some of these terms, $\Sigma_1 p^{y} \zeta'_{xy}(z) \varpi(z)^{x}$ say, are such that $x < \rho$, $y > jv$. It is possible to put

$$\Sigma_1 p^{y} \zeta'_{xy}(z) \varpi(z)^{x} = \Sigma_1 p^{y-jv} \zeta''_{xy}(z) \varpi(z)^{x} Z_r(z) + \Sigma_1' p^{y} \zeta_{xy}(z) \varpi(z)^{x}$$

where Σ_1' includes (i) terms represented by nodes on l_2, l_3, etc. and (ii) terms represented by nodes on l_1 with $\rho \leqslant x < \rho + ju$.

Other terms in (17), $\Sigma_2 p^{y} \zeta'_{xy}(z) \varpi(z)^{x}$, say, are such that $x \geqslant \rho + ju$ and again we can put

$$\Sigma_2 p^{y} \zeta'_{xy}(z) \varpi(z)^{x} = \Sigma_2 p^{y} \zeta''_{xy}(z) \varpi(z)^{x-ju} Z_r(z) + \Sigma_2' p^{y} \zeta_{xy}(z) \varpi(z)^{x},$$

Σ_2' including terms restricted as those in Σ_1'. Then

$$a_{h+1}(z) = \{p^{\sigma} \zeta_0(z) \varpi(z)^{\rho} + \Sigma_1' + \Sigma_2'\} Z_r(z) + \Sigma T_1'(z) + \Sigma T_2'(z) + \ldots$$

where the terms in $\Sigma T_r'(z)$ are represented by nodes on l_r, those included in $\Sigma T_1'(z)$ being such that $\rho \leqslant x < \rho + ju$. We next put

$$\Sigma T_1'(z) = p^\sigma \zeta_0(z)\, \varpi(z)^\rho \,.\, \Sigma_3 p^x \zeta_{xy}(z)\, \varpi(z)^y$$

with
$$vx + uy = juv + 1$$

and an appropriate adjustment of $\Sigma T_2'(z)$, etc. The number of terms in $\Sigma T_1'(z)$ may be zero but there is at least one node $(\rho + s, \sigma + t)$ on l_1 with $\rho \leqslant \rho + s < \rho + ju$. Hence follows the further adjusted expansion:

$$a_{h+1}(z) =$$
$$\left.\begin{array}{l} \{p^\sigma \zeta_0(z)\, \varpi(z)^\rho + \Sigma_1' + \Sigma_2'\}\, \{Z_r(z) + \Sigma_3 - p^t \varpi(z)^s\} \\ + p^{\sigma+t} \zeta_0(z)\, \varpi(z)^{\rho+s} + \Sigma T_2(z) + \Sigma T_3(z) + \ldots . \end{array}\right\} \quad \ldots(18).$$

37. Adding, as in § 31, suitable multiples of p^{jv+2} to the last jgu coefficients of the polynomial

$$Z_r(z) + \Sigma_3 p^g \zeta_{xy}(z)\, \varpi(z)^x - p^t \varpi(z)^s$$

we obtain an *irreducible* polynomial $X_r(z)$. The roots

$$\gamma_i, \quad i = 1, 2, \ldots jgu,$$

of $X_r(z) = 0$ are algebraic integers, and $\varpi(z)$ is the only irreducible divisor of $X_r(z) \pmod p$. Since the diagram of squares for $X_r(z)$ has only one edge, of slant v/u, each prime ideal factor of p in $[\gamma_i]$ divides p and $\varpi(\gamma_i)$ just Ku and Kv times respectively. Hence $\varpi(\gamma_i)^u/p^v$ is integral and, further,

$$N[\varpi(\gamma_i)^u/p^v] \not\equiv 0 \pmod p,$$

i.e.
$$N[\varpi(\gamma_i)] \equiv 0 \pmod{p^{jgv}} \text{ exactly} \quad \ldots\ldots\ldots\ldots(19).$$

38. Putting, after (18), $a^{(i)}{}_{h+1}(z) = a_{h+1}(z) =$

$$\{p^\sigma \zeta_0(z)\, \varpi(z)^\rho + \Sigma_1' + \Sigma_2'\}\, X_r(z) + p^{\sigma+t} \zeta_0(z)\, \varpi(z)^{\rho+s} + \Sigma T(z),$$

$\Sigma T(z)$ comprises terms represented by nodes on l_2, l_3, etc. Also since

$$\begin{aligned} a(z) &= a^{(1)}{}_{h+1}(z)\, a^{(2)}{}_{h+1}(z) \ldots a^{(w)}{}_{h+1}(z) + p^{h+1} B(z) \\ &= a^{(i)}{}_{h+1}(z)\, b^{(i)}{}_{h+1}(z) + p^{h+1} B(z), \end{aligned}$$

the norm $(-)^{njgu} N[X_r(\theta)]$

$$\begin{aligned} &= (-)^{njgu} N(\theta - \gamma_1) \ldots N(\theta - \gamma_{jgu}) \\ &= a(\gamma_1)\, a(\gamma_2) \ldots a(\gamma_{jgu}) \\ &= \prod_{1,\,jgu} [a_{h+1}(\gamma_i)\, b_{h+1}(\gamma_i) + p^{h+1} B(\gamma_i)] \\ &= \prod_{1,\,jgu} [p^{\sigma+t} b_{h+1}(\gamma_i)\, \zeta_0(\gamma_i)\, \varpi(\gamma_i)^{\rho+s} + b_{h+1}(\gamma_i)\, \Sigma T(\gamma_i) + p^{h+1} B(\gamma_i)]. \end{aligned}$$

Since $b_{h+1}(z)\, \zeta_0(z)$ is indivisible by $\varpi(z) \pmod p$, the integer

$$b_{h+1}(\gamma_i)\, \zeta_0(\gamma_i)$$

is divisible by no prime ideal factor of p in $[\gamma_i]$. Also each prime ideal factor of p in $[\gamma_i]$ divides

$$b_{h+1}(\gamma_i)\, \Sigma T(\gamma_i) + p^{h+1} B(\gamma_i)$$

more often than it divides $p^{\sigma+t}\varpi(\gamma_i)^{\rho+s}$. The norm $N[X_r(\theta)]$ is therefore just as highly divisible by p as

$$\prod_{1,\,jgu} p^{\sigma+t}\varpi(\gamma_i)^{\rho+s};$$

it therefore contains p to the exact power

$$\left.\begin{aligned} F(u, v) &= (\sigma+t)\,jgu + (\rho+s)\,jgv \\ \text{which} &= jg\,(v\rho + u\sigma + juv + 1) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(20)$$

since $(\rho+s, \sigma+t)$ lies on l_1.

39. A proof of the existence theorem stated in § 34 now follows without difficulty. The edges $E_1, E_2, \ldots E_L$ are considered in order, and the prime ideals attached to E_r contribute $j_r'u_r$, $j_r'v_r$ to the sums on the left of (15), (16). It is enough to shew that $j_r' = j_r$.

First for the edge E_1,

$$\rho = f - j_1u_1, \quad \sigma = 0, \quad F(u_1, v_1) = gj_1(v_1f + 1) \quad \ldots\ldots(21).$$

It is evident from the diagram of squares that a prime ideal attached to any of the edges $E_2, \ldots E_L$ divides $X_1(\theta)$ just as often as it divides $p^{j_1v_1}$. Further, a prime ideal attached to E_1 divides $X_1(\theta)$ just as often as the term $p^{\sigma_1}\varpi(\theta)^{\rho_1}$, represented by a node on the line

$$v_1x + u_1y = fv_1 + 1.$$

Hence, from (20) and (15),

$$(j_1u_1v_1 + 1)\,j_1'g + (j_2'u_2 + \ldots + j_L'u_L) = (fv_1 + 1)\,j_1g$$

and
$$j_1'u_1 + j_2'u_2 + \ldots + j_L'u_L = f:$$

it follows that $j_1' = j_1$, thus proving the theorem for the edge E_1.

The result for the general edge E_r follows by induction, it being assumed that

$$j_1' = j_1,\ j_2' = j_2,\ \ldots j'_{r-1} = j_{r-1}.$$

In the case of the E_r

$$\rho = f - j_1u_1 - j_2u_2 - \ldots - j_ru_r,$$

$$\sigma = j_1v_1 + j_2v_2 + \ldots + j_{r-1}v_{r-1}.$$

A prime ideal attached to any one of the edges $E_1, E_2, \ldots E_{r-1}$ divides $X_r(\theta)$ just as often as it divides $\varpi(\theta)^{j_ru_r}$, whereas one attached to

$$E_{r+1},\ E_{r+2},\ \ldots E_L$$

divides
$$X_r(\theta),\ p^{j_rv_r}$$

equally often. Each prime ideal attached to E_r, however, divides $X_r(\theta)$ just as often as it divides the term $p^{\sigma}\varpi(\theta)^{\rho}$, represented by a node on the line

$$v_rx + u_ry = j_ru_rv_r + 1.$$

It therefore follows that

$$\begin{aligned}&(j_1 v_1 + j_2 v_2 + \dots + j_{r-1} v_{r-1})\, g j_r u_r \\ &+ (j_r u_r v_r + 1)\, g j_r' + (j'_{r+1} u_{r+1} + \dots + j_L' u_L)\, g j_r v_r \\ &= F(u_r, v_r) \\ &= (j_r u_r v_r + 1)\, g j_r + (f - j_1 u_1 - j_2 u_2 - \dots - j_r u_r)\, g j_r v_r \\ &+ (j_1 v_1 + j_2 v_2 + \dots + j_{r-1} v_{r-1})\, g j_r u_r,\end{aligned}$$

i.e., on reduction, $j_r' = j_r$, the result required.

The edge E_r, whose end-nodes are $(\rho, \sigma + j_r v_r)$ and $(\rho + j_r u_r, \sigma)$, therefore attaches one or more prime ideals whose contributions to the sums in (15) and (16) are $j_r u_r$ and $j_r v_r$ respectively. When $j_r = 1$ there is only a single prime ideal, of degree g, attached to the edge, but when $j_r > 1$ there may be one or several. If, in particular,

$$j_1 = j_2 = \dots = j_L = 1$$

the prime ideal factors of $(p, \varpi(\theta))$ are completely separated by this dissection. To distinguish the various cases which arise when one of the j's exceeds unity it is necessary to dissect again.

CHAPTER VIII

THIRD DISSECTION

SEPARATION OF PRIME IDEALS ATTACHED TO THE SAME EDGE

40. The prime ideals attached to the typical edge E_r will now be called $\mathfrak{p}_r$, $\mathfrak{p}_r'$, $\mathfrak{p}_r''$, etc. Putting

$$u_r = u, \quad v_r = v, \quad j_r = j,$$

the terms represented by nodes on E_r are

$$p^\sigma \varpi(z)^\rho \zeta_0(z) \left[\varpi(z)^{ju} + p^v \zeta_1(z)\, \varpi(z)^{(j-1)u} + \dots + p^{jv} \zeta_j(z)\right] \equiv p^\sigma \varpi(z)^\rho \zeta_0(z)\, Z_r(z),$$

where $\zeta_0(z)$, $\zeta_j(z)$ are not identically divisible by p. The integers p and $\varpi(\theta)$ are respectively exactly divisible by $\mathfrak{p}_r^{\lambda u}$ and $\mathfrak{p}_r^{\lambda v}$.

41. Among the p^g integers

$$\varpi(\theta)^u + (x_1 \theta^{g-1} + x_2 \theta^{g-2} + \dots + x_g) p^v, \quad 0 \leqslant x_i < p,$$

there may or may not be one which $\equiv 0 \pmod{\mathfrak{p}_r^{\lambda uv+1}}$. There cannot however be more than one, for were

$$\varpi(\theta)^u + (x_1 \theta^{g-1} + x_2 \theta^{g-2} + \dots + x_g)\, p^v \equiv 0 \pmod{\mathfrak{p}_r^{\lambda uv+1}},$$
$$\varpi(\theta)^u + (x_1' \theta^{g-1} + x_2' \theta^{g-2} + \dots + x_g')\, p^v \equiv 0 \pmod{\mathfrak{p}_r^{\lambda uv+1}}$$

simultaneously, then would

$$(x_1 - x_1')\,\theta^{g-1} + (x_2 - x_2')\,\theta^{g-2} + \dots + x_g - x_g' \equiv 0 \pmod{\mathfrak{p}_r}$$

which is only possible when

$$x_1 - x_1' \equiv x_2 - x_2' \equiv \dots \equiv x_g - x_g' \equiv 0 \pmod{p}.$$

When none of these numbers is divisible by $\mathfrak{p}_r^{\lambda uv+1}$, there may be one of the p^{2g} integers

$$\varpi(\theta)^{2u} + (x_1\theta^{g-1} + x_2\theta^{g-2} + \dots + x_g)\,p^v\varpi(\theta)^u$$
$$+ (y_1\theta^{g-1} + y_2\theta^{g-2} + \dots + y_g)\,p^{2v}, \quad 0 \leqslant x_i < p,\ 0 \leqslant y_i < p,$$

which is divisible by $\mathfrak{p}^{2\lambda uv+1}$. Only one such integer at most, however, can be so divisible, for if

$$\varpi(\theta)^{2u} + (x_1\theta^{g-1} + \dots + x_g)\,p^v\varpi(\theta)^u + (y_1\theta^{g-1} + \dots + y_g)\,p^{2v},$$
$$\varpi(\theta)^{2u} + (x_1'\theta^{g-1} + \dots + x_g')\,p^v\varpi(\theta)^u + (y_1'\theta^{g-1} + \dots + y_g')\,p^{2v},$$

were both divisible by $\mathfrak{p}_r^{2\lambda uv+1}$, the integer

$$\{(x_1 - x_1')\,\theta^{g-2} + \dots + (x_g - x_g')\}\,\varpi(\theta)^u + \{(y_1 - y_1')\,\theta^{g-1} + \dots + (y_g - y_g')\}\,p^v$$

would be divisible by $\mathfrak{p}_r^{\lambda uv+1}$. This is certainly impossible when

$$x_1 - x_1' \equiv x_2 - x_2' \equiv \dots \equiv x_g - x_g' \equiv 0 \pmod{p},$$

and, when these congruences are not all satisfied, an integral polynomial

$$Z(z) \equiv z_1 z^{g-1} + z_2 z^{g-2} + \dots + z_g$$

can be determined to make

$$Z(z)\,\{(x_1 - x_1')\,z^{g-1} + \dots + (x_g - x_g')\} - 1,$$

divisible by $\varpi(z) \pmod{p}$ and therefore

$$\varpi(\theta)^u + Z(\theta)\,\{(y_1 - y_1')\,\theta^{g-1} + \dots + (y_g - y_g')\}\,p^v$$

divisible by $\mathfrak{p}_r^{\lambda uv+1}$ which is contrary to the assumption that

$$\varpi(\theta)^u + (x_1''\theta^{g-1} + \dots + x_g'')\,p^v$$

is indivisible by $\mathfrak{p}_r^{\lambda uv+1}$ for any integral values of x_1'', x_2'', ... x_g''.

Continuing this process a stage is eventually reached when

$$\Xi(\theta) \equiv \varpi(\theta)^{mu} + p^v\xi_1(\theta)\,\varpi(\theta)^{(m-1)u} + \dots + p^{mv}\xi_m(\theta)$$
$$\equiv \varpi(\theta)^{mu} + (x_{11}\theta^{g-1} + \dots + x_{1g})\,p^v\varpi(\theta)^{(m-1)u} + \dots$$
$$+ (x_{m1}\theta^{g-1} + \dots + x_{mg})\,p^{mv}$$
$$\equiv 0 \pmod{\mathfrak{p}_r^{m\lambda uv+1}},$$

there being no integer of the type

$$\varpi(\theta)^{su} + (x_{11}'\theta^{g-1} + \dots + x_{1g}')\,p^v\varpi(\theta)^{(s-1)u} + \dots + (x_{s1}'\theta^{g-1} + \dots + x_{sg})\,p^{sv}$$

divisible by $\mathfrak{p}_r^{s\lambda uv+1}$ with $s<m$. It follows, as above in the case where $m=2$, that

$$x_{11}, \dots x_{1g}, \dots x_{m1}, \dots x_{mg}$$

are unique residues of p. Since the p^{mg} integers

$$(y_{11}\theta^{g-1}+\dots+y_{1g})\,\varpi(\theta)^{(m-1)u}+(y_{21}\theta^{g-1}+\dots+y_{2g})\,p^v\varpi(\theta)^{(m-2)u}$$
$$+\dots+(y_{m1}\theta^{g-1}+\dots+y_{mg})\,p^{(m-1)v}, \quad 0\leqslant y_{ab}<p,$$

are all divisible by $\mathfrak{p}_r^{(m-1)\lambda uv}$ but incongruent $(\text{mod}\ \mathfrak{p}_r^{(m-1)\lambda uv+1})$, the degree of $\mathfrak{p}_r$ must be mg at least. Further, since

$$Z_r(\theta)\equiv\varpi(\theta)^{ju}+p^v\zeta_1(\theta)\,\varpi(\theta)^{(j-1)u}+\dots+p^{jv}\zeta_j(\theta)$$

is divisible by $\mathfrak{p}_r^{j\lambda uv+1}$, $m\leqslant j$.

42. The polynomial

$$\Xi(z)\equiv\varpi(z)^{mu}+p^v\xi_1(z)\,\varpi(z)^{(m-1)u}+\dots+p^{mv}\xi_m(z)$$

possesses a new kind of irreducibility. It is not congruent to the product of any two polynomials such as

$$\Xi'(z)\equiv\varpi(z)^{su}+p^v\xi_1'(z)\,\varpi(z)^{(s-1)u}+\dots+p^{sv}\xi_s'(z),$$
$$\Xi''(z)\equiv\varpi(z)^{tu}+p^v\xi_1''(z)\,\varpi(z)^{(t-1)u}+\dots+p^{tv}\xi_t''(z),$$
$$(\text{modd}\ \{\sqrt[u]{p},\ \sqrt[v]{\varpi(z)}\}^{muv+1})\equiv(\text{mod}\ W^{muv+1}),$$

where $s+t=m$, and the modulus includes all products of the type $p^\beta\varpi(z)^\alpha$ with

$$\alpha v+\beta u\geqslant muv+1.$$

For, were this decomposition possible, $\Xi(\theta)-\Xi'(\theta)\,\Xi''(\theta)$ and therefore $\Xi'(\theta)\,\Xi''(\theta)$ would $\equiv 0\ (\text{mod}\ \mathfrak{p}_r^{m\lambda uv+1})$ and so either

$$\Xi'(\theta)\equiv 0\ (\text{mod}\ \mathfrak{p}_r^{s\lambda uv+1}),$$

or

$$\Xi''(\theta)\equiv 0\ (\text{mod}\ \mathfrak{p}_r^{t\lambda uv+1}),$$

contrary to definition of m.

On dividing $Z_r(z)$ by $\Xi(z)$ there is a quotient $Q(z)$, belonging to $W^{(j-m)uv}$, and a remainder $R(z)$ of degree less than gmu and contained in W^{juv}. Putting $z=\theta$,

$$Z_r(\theta)\equiv\Xi(\theta)\,Q(\theta)+R(\theta);$$

it follows that $R(\theta)\equiv 0\ (\text{mod}\ \mathfrak{p}_r^{j\lambda uv+1})$ for $Z_r(\theta)$, $\Xi(\theta)$ are respectively divisible by $\mathfrak{p}_r^{j\lambda uv+1}$, $\mathfrak{p}_r^{m\lambda uv+1}$. Hence $R(z)$ must belong to the modulus W^{juv+1}, i.e.

$$Z_r(z)\equiv\Xi(z)\,Q(z)\ (\text{mod}\ W^{juv+1}).$$

To determine the polynomials $\Xi(z)$ it is necessary to shew that $Z_r(z)$ is congruent to a unique product of these irreducible divisors $(\text{mod}\ W^{juv+1})$ and then to shew how to find them.

43. Taking two integral polynomials

$$\beta(z) \equiv \beta_0(z)\,\varpi(z)^{bu} + p^v \beta_1(z)\,\varpi(z)^{(b-1)u} + \ldots + p^{bv}\beta_b(z),$$
$$\gamma(z) \equiv \gamma_0(z)\,\varpi(z)^{cu} + p^v \gamma_1(z)\,\varpi(z)^{(c-1)u} + \ldots + p^{cv}\gamma_c(z),$$

with $\beta_i(z) = \beta_{i1}z^{g-1} + \beta_{i2}z^{g-2} + \ldots + \beta_{ig}$, $\gamma_i(z) = \gamma_{i1}z^{g-1} + \ldots + \gamma_{ig}$, $b \geqslant c$, and carrying out the Euclidean greatest common measure process, we have

$$\left.\begin{aligned} \beta(z) - \lambda_1(z)\,\gamma(z) &\equiv p^{(b-c+1)v}\mu_1(z) \pmod{W^{buv+1}},\\ \gamma(z) - \lambda_2(z)\,\mu_1(z) &\equiv p^v \mu_2(z) \pmod{W^{cuv+1}},\\ \ldots\, \mu_{t-1}(z) - \lambda_{t+1}(z)\,\mu_t(z) &\equiv p^v \mu_{t+1}(z) \pmod{W^{(c-t+1)uv+1}} \end{aligned}\right\} (22),$$

$\lambda_1(z)$, $\mu_1(z)$, etc. being polynomials of the same type as $\beta(z)$, $\gamma(z)$ and members of the respective moduli $W^{(b-c)uv}$, $W^{(c-1)uv}$, etc. Continuation of this process leads to a stage where

$$G'(z) \equiv \mu_{t+1}(z)$$

is a divisor of $\mu_t(z) \pmod{W^{(c-t)uv+1}}$. The congruences (22), read upwards in succession, then shew that $G'(z)$ divides

$$\mu_{t-1}(z) \pmod{W^{(c-t+1)uv+1}}, \;\ldots\; \gamma(z) \pmod{W^{cuv+1}}$$

and $$\beta(z) \pmod{W^{buv+1}}.$$

Reading the same congruences downwards, they shew that any polynomial which divides

$$\beta(z) \pmod{W^{buv+1}} \text{ and } \gamma(z) \pmod{W^{cuv+1}},$$

is also a divisor of $\quad \mu_1(z) \pmod{W^{(c-1)uv+1}}$,

of $\quad \mu_2(z) \pmod{W^{(c-2)uv+1}}, \;\ldots$

and of $\quad G'(z) \pmod{W^{(c-t)uv+1}}$.

The coefficient of the highest power of $\varpi(z)$ in $G'(z)$ can be reduced to unity (as in § 35) and the modified polynomial $G(z)$ is then the highest common divisor of $\beta(z) \pmod{W^{buv+1}}$ and $\gamma(z) \pmod{W^{cuv+1}}$.

Elimination of $\mu_1(z)$, $\mu_2(z)$, … $\mu_t(z)$ from the congruences (22) puts in evidence two other polynomials

$$\beta'(z) \equiv \beta_0'(z)\,\varpi(z)^{lu} + p^v \beta_1'(z)\,\varpi(z)^{(l-1)u} + \ldots + p^{lv}\beta_v'(z),$$
$$\gamma'(z) \equiv \gamma_0'(z)\,\varpi(z)^{(b-c+l)u} + \ldots + p^{(b-c+l)v}\gamma'_{b-c+l}(z),$$

such that

$$\beta'(z)\,\beta(z) + \gamma'(z)\,\gamma(z) \equiv p^{(b+l-s)v}\,G(z)$$
$$\equiv p^{(b+l-s)v}\left[\varpi(z)^{su} + p^v\sigma_1(z)\,\varpi(z)^{(s-1)u} + \ldots + p^{sv}\sigma_s(z)\right] \pmod{W^{(b+l)uv+1}}.$$

It follows immediately that a polynomial

$$\xi(z) \equiv \varpi(z)^{mu} + p^v \xi_1(z)\,\varpi(z)^{(m-1)u} + \ldots + p^{mv}\xi_m(z),$$

irreducible $\pmod{W^{muv+1}}$, which divides

$$\beta(z)\,\gamma(z) \pmod{W^{(b+c)uv+1}},$$

must either divide $\beta(z)$ (mod W^{buv+1}) or $\gamma(z)$ (mod W^{cuv+1}). By a known argument it can be inferred that $\beta(z)$ is congruent (mod W^{buv+1}) to a unique product of irreducible divisors

$$\beta_0(z)\,\xi(z)^M \xi'(z)^{M'} \ldots,$$

the typical divisor

$$\Xi(z) \equiv \varpi(z)^{mu} + p^v \xi_1(z)\,\varpi(z)^{(m-1)u} + \ldots + p^{mv}\xi_m(z)$$

being irreducible (mod W^{muv+1}).

44. The following method is sufficient to determine the irreducible divisors of

$$\begin{aligned}\beta(z) &\equiv \varpi(z)^{bu} + p^v\beta_1(z)\,\varpi(z)^{(b-1)u} + \ldots + p^{bv}\beta_b(z)\\ &\equiv \varpi(z)^{bu} + p^v(\beta_{11}z^{g-1} + \ldots + \beta_{1g})\,\varpi(z)^{(b-1)u} + \ldots\\ &\qquad + p^{bv}(\beta_{b-1}z^{g-1} + \ldots + \beta_{bg}) \pmod{W^{buv+1}}\end{aligned}$$

in a numerical case.

Assuming that $\beta(z) \equiv \gamma(z)\,\delta(z) \pmod{W^{buv+1}}$

with $\gamma(z) \equiv \varpi(z)^{cu} + p^v\gamma_1(z)\,\varpi(z)^{(c-1)u} + \ldots + p^{cv}\gamma_c(z)$,

$$\delta(z) \equiv \varpi(z)^{(b-c)u} + p^v\delta_1(z)\,\varpi(z)^{(b-c-1)u} + \ldots + p^{(b-c)v}\delta_{b-c}(z),$$

and comparing coefficients in the product,

$$\left.\begin{aligned}\gamma_1(z) + \delta_1(z) &\equiv \beta_1(z) \pmod{p},\\ \gamma_2(z) + \delta_2(z) + \gamma_1(z)\,\delta_1(z) &\equiv \beta_2(z) \ (\mathrm{modd}\ p,\ \varpi(z))\end{aligned}\right\} \quad \ldots(23),$$

and so on.

We have already noted (§30) that the equation $\varpi(z) = 0$ is irreducible, that, if α is one of its roots, the partial basis of $[\alpha]$ is

$$(1, \alpha, \ldots \alpha^{g-1}) \ (\mathrm{bas}\ p),$$

and that (p) is a prime ideal. Putting $z = \alpha$ in (23),

$$\begin{aligned}\gamma_1(\alpha) + \delta_1(\alpha) &\equiv \beta_1(\alpha) \pmod{p},\\ \gamma_2(\alpha) + \delta_2(\alpha) + \gamma_1(\alpha)\,\delta_1(\alpha) &\equiv \beta_2(\alpha) \pmod{p},\end{aligned}$$

and so on: these imply that

$$\begin{aligned}&[w^c + \gamma_1(\alpha)\,w^{c-1} + \ldots + \gamma_c(\alpha)]\,[w^{b-c} + \delta_1(\alpha)\,w^{b-c-1} + \ldots + \delta_{b-c}(\alpha)]\\ &\qquad\equiv w^b + \beta_1(\alpha)\,w^{b-1} + \ldots + \beta_b(\alpha) \pmod{p}.\end{aligned}$$

To find the irreducible divisors of $\beta(z)$ (mod W^{buv+1}) it is therefore sufficient to determine those of

$$B(w, \alpha) \equiv w^b + \beta_1(\alpha)\,w^{b-1} + \ldots + \beta_b(\alpha) \pmod{p}$$

in the field $[\alpha]$. The symmetric product

$$B(w) \equiv B(w, \alpha^{(1)})\,B(w, \alpha^{(2)}) \ldots B(w, \alpha^{(g)})$$

has rational integral coefficients, and is congruent (mod p) to a unique product of rational integral irreducible divisors

$$B_1(w)\,B_2(w)\,\ldots\,B_k(w).$$

Each such divisor $B_i(w)$ has a divisor in $[\alpha]$ in common with $B(w, \alpha)$ (mod p), and this can be calculated by the greatest common measure process.

45. The irreducible divisors of $Z_r(z)$ (mod W^{juv+1}) will now be taken to be

$$\Xi(z)^M\,\Xi'(z)^{M'}\,\Xi''(z)^{M''}\,\ldots,$$

with $$\Xi(z) = \varpi(z)^{mu} + p^v\xi_1(z)\,\varpi(z)^{(m-1)u} + \ldots + p^{mv}\xi_m(z),$$

and $$Mm + M'm' + M''m'' + \ldots = j.$$

A prime ideal $\mathfrak{p}_r$ attached to E_r divides just one of the integers $\Xi(\theta)$, $\Xi'(\theta)$, $\Xi''(\theta)$, etc. more often than it divides the several terms of which that number is the sum. If the number in question is $\Xi(\theta)$, $\Xi(\theta)$ and $\mathfrak{p}_r$ will be said to *annex* each other.

Conversely it can be shewn that each of the factors $\Xi(\theta)$, $\Xi'(\theta)$, $\Xi''(\theta)$, etc. annexes at least one of the prime ideals attached to the edge.

The expansion

$$a_{h+1}(z) = p^\sigma\zeta_0(z)\,\varpi(z)^\rho\,\Xi(z)^M\,\Xi'(z)^{M'}\,\Xi''(z)^{M''}\,\ldots$$

+ terms represented by nodes on l_1, l_2, etc. is first modified, as in § 35, to the form

$$a_{h+1}(z) = K(z)\,\Xi(z)^M\,\Xi'(z)^{M'}\,\Xi''(z)^{M''}\,\ldots + \Sigma T_1'(z) + \Sigma T_2'(z) + \ldots$$

where $$\Sigma T_1'(z) \equiv \Sigma p^y\xi_{xy}(z)\,\varpi(z)^x, \quad \rho \leqslant x < \rho + ju,$$

contains all the terms amongst $\Sigma T_1'(z) + \Sigma T_2'(z) + \ldots$ represented by nodes on l_1, and

$$K(z) = p^\sigma\zeta_0(z)\,\varpi(z)^\rho + \Sigma p^{\nu - jv}\zeta_1''(z)\,\varpi(z)^\mu + \Sigma p^{\nu'}\zeta_2''(z)\,\varpi(z)^{\mu' - ju}.$$

A further modification on similar lines brings it to the form

$$a_{h+1}(z) = K(z)\,[\Xi(z)^M\,\Xi'(z)^{M'}\,\Xi''(z)^{M''}\,\ldots + \Sigma_0 p^y\beta(z)\,\varpi(z)^x]$$
$$+ T_1(z) + \Sigma T_2''(z),$$

there being only one term, $T_1(z)$, in the second line represented by a node on l_1. Each term under Σ_0 belongs to W^{juv+1} but not to W^{juv+2}. Since the polynomials

$$\Xi(z)^M, \quad \Xi'(z)^{M'}, \quad \Xi''(z)^{M''}, \quad \ldots$$

have no common divisor in the sense of § 43, it follows without difficulty that there exists a further set of polynomials

$$H(z),\quad H'(z),\quad H''(z),\quad \dots$$

belonging to the respective moduli

$$W^{Mmuv+1},\quad W^{M'm'uv+1},\quad W^{M''m''uv+1},\quad \dots$$

and such that

$$\Sigma\left[\frac{H(z)}{\Xi(z)^M}\,\Pi\,\Xi(z)^M\right] \equiv \Sigma_0 p^y \beta(z)\,\varpi(z)^x \pmod{W^{juv+2}}.$$

The expansion therefore admits the further adjustment

$$a_{h+1}(z) = K(z)\,\Pi\,\{\Xi(z)^M + H(z)\} + T_1(z) + \Sigma T_2'''(z) + \dots.$$

Finally, adding suitable multiples of p^{Mmuv+2} to the last $Mmgu$ coefficients of $\Xi(z)^M + H(z)$ an irreducible polynomial $\Xi_1(z)$ is obtained and

$$a_{h+1}(z) = K(z)\,\Xi_1(z)\,\Pi'\,\{\Xi'(z)^{M'} + H'(z)\} + T_1(z) + \Sigma T_2(z) + \dots.$$

46. The roots $\epsilon_1, \epsilon_2, \dots \epsilon_{Mmgu}$ of $\Xi_1(\epsilon) = 0$ are algebraic integers: also, as in § 37, $\varpi(\epsilon_i)^u/p^v$ is integral and

$$N[\varpi(\epsilon_i)] \equiv 0 \pmod{p^{Mmgv}}$$

exactly.

Next to examine the divisibility by p of

$$(-)^{Mmngu} N[\Xi_1(\theta)],$$

which

$$\begin{aligned} &= (-)^{Mmngu} N(\theta - \epsilon_1)\,N(\theta - \epsilon_2) \dots N(\theta - \epsilon_{Mmgu}) \\ &= a(\epsilon_1)\,a(\epsilon_2) \dots a(\epsilon_{Mmgu}) \\ &= \prod_{1,\,Mmgu} [\{T_1(\epsilon_i) + \Sigma T_2(\epsilon_i)\}\,Q(\epsilon_i) + p^{Mmv+1} R(\epsilon_i)]. \end{aligned}$$

$Q(z)$ being indivisible by $\varpi(z)$ (mod p), p divides this last just as often as

$$\prod_{1,\,Mmgu} T_1(\epsilon_i) \text{ or } \prod_{1,\,Mmgu} p^y \varpi(\epsilon_i)^x$$

where

$$yu + xv = \rho v + \sigma u + juv + 1.$$

Hence

$$N[\Xi_1(\theta)] \equiv 0 \pmod{p^{Mmg(\rho v + \sigma u + juv + 1)}} \text{ exactly} \dots\dots\dots (24).$$

47. A prime ideal factor of $(p, \varpi(\theta))$, attached to one of the edges $E_1, E_2, \dots E_{r-1}$, divides $\Xi_1(\theta)$ just as often as it divides $\varpi(\theta)^{Mmu}$, whereas one attached to $E_{r+1}, E_{r+2}, \dots E_L$ divides $\Xi_1(\theta)$ and p^{Mmv} equally often. Prime ideals attached to E_r but not annexed by $\Xi(\theta)$ divide $\Xi_1(\theta)$ as often as $\Xi(\theta)^M$ or $\varpi(\theta)^{Mmu}$. Finally a prime ideal annexed by $\Xi(\theta)$ divides $\Xi_1(\theta)$ as often as $p^y \varpi(\theta)^x$ where

$$uy + vx = Mmuv + 1.$$

Take Cu, Cv to be the contributions of the prime ideals annexed by $\Xi(\theta)$ to the sums of the left of (15), (16). Then, equating $N[\Xi(\theta)]$, as in (24), to the norm of the product of its prime ideal factors,

$$\begin{aligned} &Mmg(\rho v + \sigma u + juv + 1) \\ &\quad = Mm\sigma gu \text{ (due to prime ideals attached to } E_1, \dots E_{r-1}) \\ &\quad + Mm\rho gv \text{ (due to prime ideals attached to } E_{r+1}, \dots E_L) \\ &\quad + C(Mmuv+1)g \text{ (due to prime ideals annexed by } \Xi(\theta)) \\ &\quad + (j-C)Mmguv \text{ (due to prime ideals attached to } E_r \text{ but not} \\ &\qquad\quad \text{annexed by } \Xi(\theta)). \end{aligned}$$

It follows that $$C = Mm \qquad\qquad (25).$$

Every irreducible divisor of $Z_r(\theta)$ (mod W^{juv+1}) therefore annexes at least one prime ideal attached to the edge.

In the particular case when all the indices

$$M, M', M'', \dots$$

are equal to unity the third dissection completely separates the prime ideals attached to E_r. The divisor $\Xi(\theta)$ then annexes a prime ideal of degree mg, $\Xi'(\theta)$ a different prime ideal of degree $m'g$, etc., and

$$m + m' + m'' + \dots = j.$$

When some of the indices

$$M, M', M'', \dots$$

exceed unity the third dissection is insufficient to separate the prime ideal factors of $(p, \varpi(\theta))$ completely. It can be proved however, just as in § 29, that the degree of every prime ideal annexed by $\Xi(\theta)$ is a multiple of mg. Further to distinguish the cases that can arise a fourth dissection is necessary (*v.* Preface and Chap. XIV).

CHAPTER IX

CONSTRUCTION OF THE PARTIAL BASIS OF $\mathfrak{o}$ (bas p).

48. The two lines $y = k-1$, $y = k$ (k integral) cross the boundary at $(\mu, k-1)$, (λ, k), two points on E_r, and $\nu (= \nu_k)$, π are the integers defined by

$$0 \leqslant \lambda - \nu < 1, \quad 0 \leqslant \mu - \pi < 1.$$

Evidently $$\lambda < \mu, \quad \nu \leqslant \pi.$$

The terms in (9) represented by nodes between $y=0$ and $y=k$ 1 inclusive, by nodes on $y=k$, and by other nodes are

$$\varpi(z)^{\pi} X_k(z), \quad \varpi(z)^{\nu} X'_k(z), \quad X''_k(z)$$

respectively, so that

$$a_{h+1}(z)=\varpi(z)^{\pi} X_k(z)+\varpi(z)^{\nu} X'_k(z)+X''_k(z).$$

49. It is necessary to examine the divisibility of $X_k(\theta)$ by a prime ideal $\mathfrak{p}_s$ attached to E_s, the cases $s<r$, $s>r$ and $s=r$ needing separate discussion.

(i) $s<r$. When (ν, k) is a marked node, it is evident from the diagram that $\mathfrak{p}_s$ divides $\varpi(\theta)^{\pi} X_k(\theta)$ just as often as, and

$$\varpi(\theta)^{\pi} X_k(\theta)+p^k \zeta_{\nu k}(\theta)\, \varpi(\theta)^{\nu}$$

more often than, $p^k \varpi(\theta)^{\nu}$, and when (ν, k) is unmarked, $\mathfrak{p}_s$ divides $\varpi(\theta)^{\pi} X_k(\theta)$ more often than $p^k \varpi(\theta)^{\nu}$.

Hence, in either event, $\mathfrak{p}_s$ divides $\varpi(\theta)^{\pi-\nu} X_k(\theta)$ at least as often as p^k.

(ii) $s>r$. Again the diagram shews that $\mathfrak{p}_s$ divides $\varpi(\theta)^{\pi} X_k(\theta)$ at least as often as $p^{k-1}\varpi(\theta)^{\mu}$. This prime ideal also divides $\varpi(\theta)^{\mu-\lambda}$ more often than p. It therefore divides $\varpi(\theta)^{\pi+\mu-\lambda} X_k(\theta)$ more often than $p^k \varpi(\theta)^{\mu}$. Hence, since $\lambda \geqslant \nu$, $\mathfrak{p}_s$ divides $\varpi(\theta)^{\pi-\nu} X_k(\theta)$ more often than p^k.

(iii) $s=r$. A prime ideal $\mathfrak{p}_r$ attached to E_r divides $\varpi(\theta)^{\pi} X_k(\theta)$ at least as often as $p^k \varpi(\theta)^{\lambda}$—represented by a marked node on E_r. Hence, since $\nu \leqslant \lambda$, $\mathfrak{p}_r$ divides $\varpi(\theta)^{\pi-\nu} X_k(\theta)$ at least as often as p^k.

[Whenever $\nu<\lambda$, and in certain cases when $\nu=\lambda$, $\mathfrak{p}_r$ divides

$$\varpi(\theta)^{\pi-\nu} X_k(\theta)$$

more often than p^k.]

50. It will now be assumed, up to the end of Chap. x, that the third dissection completely separates the prime ideal factors of $(p, \varpi(\theta))$, i.e. that all the indices $M, M', M'', \ldots$, for every edge of the boundary, are equal to unity. Subject to this assumption the polynomial

$$\varpi(\theta)^{\pi-\nu} X_k(\theta)$$

will be identified with $\chi_k(\theta)$ of § 24. It has just been seen that every prime ideal factor of $(p, \varpi(\theta))$ divides $\varpi(\theta)^{\pi-\nu} X_k(\theta)$ at least as often as p^k. To establish the identification it is sufficient to prove that there exists no integral polynomial

$$C(\theta) \equiv \theta^{c}+c_1\theta^{c-1}+\ldots+c_c,$$

of lower rank than $\varpi(\theta)^{\pi-\nu} X_k(\theta)$, and divisible by each prime ideal factor of $(p, \varpi(\theta))$ as often as p^k.

The polynomial $C(z)$ can be expanded in the form

$$C(z) \equiv \eta_q(z)\,\varpi(z)^q + \eta_{q-1}(z)\,\varpi(z)^{q-1} + \ldots + \eta_0(z)$$

where the integral polynomials $\eta_q(z), \ldots, \eta_0(z)$ are of degree $g-1$ or lower. Further

$$\eta_i(z) = \xi_{i0}(z) + p\,\xi_{i1}(z) + \ldots + p^{k-1}\xi_{i\,k-1}(z) + p^k \xi_{ik}(z),$$

the coefficients of $\xi_{i0}(z), \ldots, \xi_{i\,k-1}(z)$ being less than p and not negative. Evidently (mod p^k),

$$\left.\begin{aligned} C(\theta) &\equiv C_0(\theta) \\ &= \xi_{q0}(\theta)\,\varpi(\theta)^q + \xi_{q-10}(\theta)\,\varpi(\theta)^{q-1} + \ldots + \xi_{10}(\theta)\,\varpi(\theta) + \xi_{00}(\theta) \\ &\quad + p[\xi_{q1}(\theta)\,\varpi(\theta)^q + \xi_{q-11}(\theta)\,\varpi(\theta)^{q-1} + \ldots + \xi_{11}(\theta)\,\varpi(\theta) + \xi_{01}(\theta)] \\ &\quad + \ldots + p^{k-1}[\xi_{q\,k-1}(\theta)\,\varpi(\theta)^q + \ldots + \xi_{1\,k-1}(\theta)\,\varpi(\theta) + \xi_{0\,k-1}(\theta)] \end{aligned}\right\}(26).$$

Since $\varpi(\theta)^{\pi-\nu} X_k(\theta) = \varpi(\theta)^{f-\nu} +$ terms of lower rank, it will suffice to prove that $q \geqslant f-\nu$ when $C_0(\theta)$ contains every prime ideal factor of $(p, \varpi(\theta))$ as often as p^k.

The terms $\xi_{ij}(\theta)$ are not usually all present in (26). With a modified suffix notation the terms divisible least often by prime ideals attached to E_s are

$$\left.\begin{aligned} &p^{b_s}\varpi(\theta)^{a_s}[\xi_0(\theta)\,\varpi(\theta)^{t_s u_s} + \ldots + p^{t_s v_s}\xi_{t_s}(\theta)] \\ \equiv\, &p^{b_s}\varpi(\theta)^{a_s}\,T^s(\theta) \end{aligned}\right\} \ldots\ldots\ldots\ldots(27),$$

and represented by nodes on a line E_s' parallel to E_s. Plotting all the nodes representing terms occurring in (26) and forming a boundary as in § 33, E_s' is an edge of this boundary: this diagram will be referred to as the c-diagram and the former one as the a-diagram when it is necessary to distinguish between them.

Every term in (27) contains each of the prime ideals $\mathfrak{p}_s$, $\mathfrak{p}_s'$, $\mathfrak{p}_s''$, …, just $b_s u_s + a_s v_s + t_s u_s v_s$ times, and the whole expression (27) can only contain each of these prime ideals more often than this when $t_s \geqslant j_s$ (i.e. when E_s' is at least as long as E_s).

51. First take $r=1$, whence necessarily $1 \leqslant k \leqslant j_1 v_1$. Each of the prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_1'$, …, divides $\varpi(\theta)^{f-\nu_k}$ at least as often as, but $\varpi(\theta)^{f-\nu_k-1}$ less often than, p^k. Further, when $t < j_1$, the whole expression (27) is divisible by at least one of these prime ideals no more often than one of its terms, and each term therein contains $\mathfrak{p}_1$ to the power

$$b_1 u_1 + a_1 v_1 + t u_1 v_1$$

which (from the diagram) $\leqslant qv_1$. Hence, when $C(\theta)$ is divisible by all the prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_1'$, ..., as often as p^k, q cannot be less than $f-\nu_k$. It is therefore sufficient to put

$$\chi_k(\theta)=\varpi(\theta)^{f-\nu_k} \text{ with } 1\leqslant k\leqslant j_1v_1$$

or, equally well,

$$\chi_k(\theta)=\varpi(\theta)^{\pi-\nu_k}X_k(\theta),$$

since evidently the prime ideal factors of $(p, \varpi(\theta))$ attached to E_2, E_3 etc. divide either of these numbers more often than p^k.

52. Next when $r>1$ the last non-vanishing coefficient in the first row of (26) will be taken to be $\xi_{q'0}(\theta)$: clearly $q'\leqslant q$. Produced if necessary the line E_1' cuts the ϖ-axis at $(q'', 0)$ where $q''\leqslant q'\leqslant q$. Thus the several terms in $p^{b_1}\varpi(\theta)^{a_1}T_1(\theta)$ are, at most, as highly divisible as $\varpi(\theta)^q$ by prime ideals attached to E_1. An inspection of the diagram shews that each of these prime ideals divides p^k more often than $\varpi(\theta)^{f-\nu_1}$, and therefore than $\varpi(\theta)^q$ if q be supposed less than $f-\nu_1$. Hence $T_1(\theta)$ is more highly divisible by $\mathfrak{p}_1$, $\mathfrak{p}_1'$, ..., than one of the terms whose aggregate it is: whence $T_1(z)$ is divisible by every irreducible divisor of $Z_1(z)$ (mod $W^{j_1u_1v_1+1}$). Since these irreducible divisors are all distinct, the edge E_1' must be at least as long as E_1, and the length of its projection on the ϖ-axis be j_1u_1 units at least.

I now assume that the c-diagram has been proved to contain edges E_1', E_2', ..., E'_{s-1} $(s<r)$ parallel to, and at least as long as, E_1, E_2, ..., E_{s-1} respectively, and pass to the next stage by induction.

It is necessary to prove that the expansion (26) contains at least one term less highly divisible than p^k by prime ideals attached to E_s. Since no two edges of the c-diagram can cross each other it is readily seen that the node

$$(q-e,\ c')\equiv(q-j_1u_1-\ldots-j_{s-1}u_{s-1},\ j_1v_1+\ldots+j_{s-1}v_{s-1}),$$

either lies on its boundary or else on the side of it opposite to the origin. So if $(q-e,\ c_0)$ lies on the boundary, $c_0\leqslant c'$. By definition of ν_k as a function of k the point $(\nu_k+1,\ k')$, with $k'<k$, lies on the boundary of the a-diagram: also a line drawn through $(q-e,\ c')$ parallel to E_s passes through $(\nu_k+1,\ k'')$ with $k''\leqslant k'$ (the inequality only reducing to an equality when $s=r$). Further, since q is supposed less than $f-\nu_k$, a line through $(q-e,\ c_0)$ cuts the p-axis at $(0,\ K)$ where

$$K\leqslant k'' \text{ which } \leqslant k' \text{ which } <k.$$

A prime ideal attached to E_s divides $p^c \varpi(\theta)^{q-e}$ just as often as p^k, and at least as often as those terms in (26) which it divides least often. Hence the expansion (26) contains at least one term less highly divisible than p^k by $\mathfrak{p}_s$, $\mathfrak{p}_s'$ etc., and this is the result stated.

The integer $T_s(\theta)$ is now proved to be more highly divisible by $\mathfrak{p}_s$, $\mathfrak{p}_s'$ etc. than the several terms of which it is the sum. If each separate term $\equiv 0 \pmod{W^w}$ exactly it is only when $w \geqslant j_s u_s v_s$ that this statement can be true. Accordingly E_s' is at least as long as E_s.

53. No two edges of the c-boundary cross each other, neither do the projections on the ϖ-axis of any two edges overlap. The length of the projection of the c-boundary is therefore

$$q''' \equiv j_1 u_1 + \dots + j_r u_r$$

units at least, and $q''' < q$. Also, from definition of ν_k,

$$f - \nu_k \leqslant j_1 u_1 + \dots + j_r u_r,$$

which shews that the former inequality is inconsistent with the assumption $q < f - \nu_k$.

Hence, subject only to the condition that the prime ideals dividing $(p, \varpi(\theta))$ are entirely separated by the third dissection, there is no integral polynomial $c(\theta)$ of lower rank than $\varpi(\theta)^{\pi-\nu} X_k(\theta)$ so highly divisible as p^k by all these prime ideals. It follows that

$$\chi_k^{(i)}(\theta) = \dot{\chi}_k(\theta) = \varpi(\theta)^{\pi-\nu} X_k(\theta)$$

for $$k \leqslant j_1 v_1 + j_2 v_2 + \dots + j_i v_i = h.$$

When $k \geqslant h$, $\chi_k(\theta)$ is of rank fg. Since

$$a(z) = a^{(1)}{}_k(z) \dots a^{(w)}{}_k(z) + p^k \nu_k(z),$$

it is sufficient to take $\chi_k(\theta) = a_k(\theta)$ for these values of k.

CHAPTER X

THE BASES OF A PRIME IDEAL

54. It will now be assumed that the stem of the partial basis of $\mathfrak{o}$ (bas p) has been obtained, as in Chap. IX, for each prime factor of $D(\theta)$, and that the several partial bases have been combined, as in Chap. IV, to give the integral basis in the form

$$\mathfrak{o} \equiv (\omega_0, \omega_1, \dots \omega_{n-1}) \equiv \left(1, \frac{\Psi_1(\theta)}{\Delta_1}, \dots \frac{\Psi_{n-1}(\theta)}{\Delta_{n-1}}\right).$$

An ideal $\mathfrak{j}$ of $[\theta]$ is expressible as a compound modulus of order n,

$$\mathfrak{j} \equiv (\alpha_0, \alpha_1, \dots \alpha_{n-1})$$

with α_r an integer of rank r, i.e. it is the aggregate of integers of the type $l_0 \alpha_0 + l_1 \alpha_1 + \dots + l_{n-1} \alpha_{n-1}$ for rational integral values of $l_0, l_1, \dots l_{n-1}$.

When $\mathfrak{j}$ is a prime ideal $\mathfrak{p}$, $\alpha_0 = p$, a rational prime number, and every member of the principal ideal

$$p\,\mathfrak{o} \equiv (p\omega_0, p\omega_1, \dots p\omega_{n-1})$$

also belongs to $\mathfrak{p}$. Among the integers of rank r contained in $\mathfrak{p}$, viz.: $n_0\omega_0 + n_1\omega_1 + \dots + n_r\omega_r$, there may or may not be one with $n_r = 1$. In the former event it is enough to put

$$\alpha_r = n_0\omega_0 + n_1\omega_1 + \dots + \omega_r,$$

in the latter event $\alpha_r = p\omega_r$.

The norm of the prime ideal is $N(\mathfrak{p}) \equiv p^d$, the number of incongruent residues of $\mathfrak{p}$ in $\mathfrak{o}$. Clearly d is the number of values of r for which $\alpha_r = p\omega_r$; also, in the present notation, $d = mg$.

The bases of some special types of prime ideal will be found before mention of the general case § 58.

55. If p is prime to $\Delta_1, \Delta_2, \dots \Delta_{n-1}$

$$\mathfrak{o} \equiv (1, \theta, \dots \theta^{n-1}) \text{ (bas } p)$$

and

$$a(z) \equiv \varpi_1(z)^{f_1} \dots \varpi_w(z)^{f_w} \pmod{p}.$$

There being no complex integer $\phi(\theta)/p$ of rank $< n$ the $\varpi_i(z)$-diagram only contains one edge, joining $(f_i, 0)$ to $(0, 1)$ or else $(1, 0)$ to $(0, h_i)$. Accordingly (p) contains the ideal factor $\mathfrak{p}_i^{f_i}$, $\mathfrak{p}_i$ being prime and of degree g_i; in fact

$$(p) = \mathfrak{p}_1^{f_1}\,\mathfrak{p}_2^{f_2} \dots \mathfrak{p}_w^{f_w}$$

and, with an obvious extension of the previous notation,

$$\mathfrak{p}_i \equiv (p, p\theta, \dots p\theta^{g_i-1}, \varpi_i(\theta), \theta\varpi_i(\theta), \dots \theta^{n-1-g_i}\varpi_i(\theta)) \text{ (bas } p).$$

This modulus is to be called the partial basis of $\mathfrak{p}_i$ (bas p) and $(p, \varpi_i(\theta))$ its stem.

In general a prime ideal $\mathfrak{p} \equiv (\beta_0, \beta_1, \dots \beta_{n-1})$ (bas p) when no complex integer of the type $(m_0\beta_0 + m_1\beta_1 + \dots + \beta_r)/p$ belongs to $\mathfrak{p}$.

Since, in the present case, $dv(p, \Delta_r) = 1$ there exists a polynomial $\Psi_r(z)$ of degree r and with unit leading coefficient such that

$$\left.\begin{aligned} &\Psi_r(z) \equiv z^{r-g_i}\varpi_i(z) \pmod{p} \\ \text{and}\quad &\Psi_r(z) \equiv \psi_r(z) \pmod{\Delta_r} \end{aligned}\right\} r = g_i, \dots n-1.$$

A basis of $\mathfrak{p}_i$ is then

$$\left(p, \frac{p\,\Psi_1(\theta)}{\Delta_1}, \dots \frac{p\,\Psi_{g_i-1}(\theta)}{\Delta_{g_i-1}}, \frac{\Psi_{g_i}(\theta)}{\Delta_{g_i}}, \dots \frac{\Psi_{n-1}(\theta)}{\Delta_{n-1}}\right).$$

56. Next when one of the w-diagrams, that for $\varpi(z)$ say, contains only one edge, joining $(f, 0)$ to $(0, 1)$, there is one prime ideal $\mathfrak{p}$, of degree g, which divides $\varpi(\theta)$ and p just once and f times respectively. The stem of $\mathfrak{o}$ (bas p) is

$$1, \frac{\chi_1(\theta)\,U_1(\theta)}{p}, \frac{\chi_2(\theta)\,U_2(\theta)}{p^2}, \dots \frac{\chi_l(\theta)\,U_l(\theta)}{p^l}$$

where $\chi_r(z) \equiv \varpi(z)^f \pmod{p}$ and $U_r(\theta)$ is due to the irreducible divisors $\varpi_1(z)$, $\varpi_2(z)$ etc., excluding $\varpi(z)$, of $a(z) \pmod{p}$. Since $\mathfrak{p}^{rf}$ divides $\chi_r(\theta)$ the number $\chi_r(\theta)\, U_{r-1}(\theta)/\mathfrak{p}^{r-1}$ is an integer divisible by $\mathfrak{p}$: accordingly the stem of $\mathfrak{p}$ (bas p) is

$$\left(p,\ \varpi(\theta),\ \frac{\chi_2(\theta)\, U_1(\theta)}{p},\ \ldots\ \frac{\chi_{l+1}(\theta)\, U_l(\theta)}{p^l}\right).$$

57. A more general case occurs when the $\varpi(z)$-diagram only contains one edge, of slant v/u with $f = ju$. The prime ideal $\mathfrak{p}$ annexed by the irreducible divisor

$$\Xi(z) \equiv \varpi(z)^{mu} + p^v \xi_1(z)\, \varpi(z)^{(m-1)v} + \ldots + p^{mv} \xi_m(z)$$

of

$$Z(z) \equiv \varpi(z)^{ju} + p^v \zeta_1(z)\, \varpi(z)^{(j-1)u} + \ldots + p^{jv} \zeta_j(z)$$

(mod W^{juv+1}) is of degree mg. Defining $U_r(\theta)$ as in § 56 each of the complex integers

$$\frac{\chi_r(\theta)\, U_r(\theta)}{p^r}, \quad 1 \leqslant r \leqslant jv - 1, \quad r \not\equiv 0 \pmod{v},$$

is divisible by $\mathfrak{p}$. Further, a set of integral polynomials

$$\lambda_{r1}(z), \ldots \lambda_{rm}(z), \quad r = mv,\ (m+1)v, \ldots (j-1)v,$$

of degree $g - 1$ or lower, can be found to make

$$\chi_r'(z) \equiv \chi_r(z) + p^v \lambda_{r1}(z)\, \chi_{r-v}(z) + \ldots + p^{mv} \lambda_{rm}(z)\, \chi_{r-mv}(z)$$

divisible by $\Xi(z)$(mod W^{ruv+1}). The numbers

$$\frac{\chi'_{mv}(\theta)\, U_{mv}(\theta)}{p^{mv}},\ \frac{\chi'_{(m+1)v}(\theta)\, U_{(m+1)v}(\theta)}{p^{(m+1)v}},\ \ldots\ \frac{\chi'_{(j-1)v}(\theta)\, U_{(j-1)v}(\theta)}{p^{(j-1)v}}$$

are then complex integers divisible by $\mathfrak{p}$.

It follows that the stem of $\mathfrak{p}$ (bas p) is

$$\left.\begin{aligned}
& p,\ \varpi(\theta),\ \frac{\chi_1(\theta)\, U_1(\theta)}{p},\ \ldots\ \frac{\chi_{v-1}(\theta)\, U_{v-1}(\theta)}{p^{v-1}},\ \frac{p\,\chi_v(\theta)\, U_v(\theta)}{p^v},\\
\ldots\ & \frac{\chi_{(m-2)v+1}(\theta)\, U_{(m-2)v+1}(\theta)}{p^{(m-2)v+1}},\ \ldots\ \frac{p\,\chi_{(m-1)v}(\theta)\, U_{(m-1)v}(\theta)}{p^{(m-1)v}},\\
& \frac{\chi_{(m-1)v+1}(\theta)\, U_{(m-1)v+1}(\theta)}{p^{(m-1)v+1}},\ \ldots\ \frac{\chi'_{mv}(\theta)\, U_{mv}(\theta)}{p^{mv}},\\
\ldots\ & \frac{\chi_{(j-2)v+1}(\theta)\, U_{(j-2)v+1}(\theta)}{p^{(j-2)v+1}},\ \ldots\ \frac{\chi'_{(j-1)v}(\theta)\, U_{(j-1)v}(\theta)}{p^{(j-1)v}},\\
& \frac{\chi_{(j-1)v+1}(\theta)\, U_{(j-1)v+1}(\theta)}{p^{(j-1)v+1}},\ \ldots\ \frac{\chi_{jv+1}(\theta)\, U_{jv}(\theta)}{p^{jv}},\\
& \frac{\chi_{jv+2}(\theta)\, U_{jv+1}(\theta)}{p^{jv+1}},\ \ldots\ \frac{\chi_{l+1}(\theta)\, U_l(\theta)}{p^l},
\end{aligned}\right\} \quad (28)$$

the first line containing $v+2$ terms and the next $j-1$ lines v terms each. To insert the last $l-jv+1$ terms it is noted that $\chi_k(\theta)$ is of rank gju whenever $k \geqslant jv$.

58. For the most general prime ideal isolated by the third dissection the $(p, \varpi(z))$-diagram has several distinct edges and $\mathfrak{p}$ is attached to an edge E_r whose end-nodes are $(\rho+ju, \sigma)$, $(\rho, \sigma+jv)$. Marked nodes on E_r represent the terms $p^\sigma \varpi(z)^\rho \zeta_0(z) Z_r(z)$, and $\mathfrak{p}$, a prime ideal of degree mg, annexes the irreducible divisor $\Xi(z)$ of $Z_r(z) \pmod{W^{juv+1}}$. The first two terms of the stem of $\mathfrak{p}$ (bas p) are p, $\varpi(\theta)$: its remaining terms are obtained from the stem of $\mathfrak{o}$ (bas p) by changing $\chi_k(\theta)$ into a number $\chi_k'(\theta)$ defined in the following way.

(i) $\chi_k'(\theta) = \chi_k(\theta)$ or $\chi_k(\theta) + p^k \zeta_{\nu k}(\theta)$, according as (ν, k) is unmarked or marked, when $1 \leqslant k \leqslant \sigma$. [§ 49, i.]

(ii) $\chi_k'(\theta) = \chi_k(\theta)$ when

$$k = \sigma + rv + s \text{ and } 0 \leqslant r \leqslant j-1,\ 1 \leqslant s \leqslant v-1. \qquad [\S\ 49,\ \text{iii.}]$$

(iii) $\chi_k'(\theta) = p\chi_k(\theta)$ when

$$k = \sigma + rv,\ 1 \leqslant r \leqslant m-1.$$

(iv) When $k = \sigma + rv$, $m \leqslant r \leqslant j-1$, a set of polynomials

$$\lambda_{r1}(z),\ \lambda_{r2}(z),\ \dots\ \lambda_{rm}(z),$$

of degree $g-1$ or lower, can be chosen to make

$$\chi_{k0}(z) \equiv \chi_k(z) + p^v \lambda_{r1}(z) \chi_{k-v}(z) + \dots + p^{mv} \lambda_{rm}(z) \chi_{k-mv}(z)$$

divisible by $\Xi(z) \pmod{W^{\rho v+ku+1}}$. In these cases

$$\chi_k'(\theta) = \chi_{k0}(\theta).$$

(v) $\chi'_{\sigma+jv}(\theta) = \chi_{\sigma+jv}(\theta) + p^{\sigma+jv} \zeta_{\rho\, \sigma+jv}(\theta)$.

(vi) $\chi_k'(\theta) = \chi_k(\theta)$ when $\sigma + jv < k < h$. (§ 49, ii.)

(vii) $\chi_k'(\theta) = \chi_{k+1}(\theta)$ when $k \geqslant h$.

59. When several successive members of the stem are of equal rank, only the last one is to be retained. If Ω, Ω' of respective ranks s, t, are consecutive terms of the stem thus modified, the terms succeeding Ω in the partial basis of $\mathfrak{p}$ (bas p) are

$$\Omega\theta,\ \Omega\theta^2,\ \dots\ \Omega\theta^{g-1},\ \frac{\Omega\varpi(\theta)}{p},\ \frac{\Omega\theta\varpi(\theta)}{p},\ \dots\ \frac{\Omega\theta^{t-s-g-1}\varpi(\theta)}{p}$$

when Ω is of rank 0 or $\sigma+rv$, $1 \leqslant r \leqslant m-1$, and $\Omega\theta$, $\Omega\theta^2$, $\dots\ \Omega\theta^{t-s-1}$ in all other cases. To obtain a complete basis of $\mathfrak{p}$ such other prime factors of $D(\theta)$ as admit complex integers $\phi'(\theta)/p'$ are introduced into the denominators as in § 55 above.

It is to be observed that in §§ 56, 57, 58 the expansion of Chapter V is required to one more stage than for the calculation of the stem of $\mathfrak{o}$ (bas p).

CHAPTER XI

NUMERICAL ILLUSTRATIONS

60. I. $$a(\theta) \equiv \theta^4 - 8\theta^3 + 6\theta^2 - 24\theta - 108 = 0.$$

The invariants of the quartic are $I = -153$, $J = 311$,

$$D(\theta) = 256(I^3 - 27J^2) = -2^{10}.3^3.11.13.401.$$

The highest common factor of $a(z)$, $a'(z)$ (mod 13) is $z+9$ whence, without difficulty,

$$a(z) \equiv (z+9)^2 (z+6)(z+7)$$

and $(13) = \mathfrak{p}_{13}{}^2 \mathfrak{q}_{13} \mathfrak{r}_{13}$ with

$$\begin{aligned} \mathfrak{p}_{13} &\equiv (13,\ \theta+9,\ \theta^2+10,\ \theta^3+1)\ (\text{bas } 13), \\ \mathfrak{q}_{13} &\equiv (13,\ \theta+6,\ \theta^2+3,\ \ \theta^3+8)\ (\text{bas } 13), \\ \mathfrak{r}_{13} &\equiv (13,\ \theta+7,\ \theta^2+3,\ \ \theta^3+5)\ (\text{bas } 13). \end{aligned}$$

Again $$a(z) \equiv z^3 (z+1)\ (\text{mod } 3),$$

and the expansion of Chap. V is

$$a(z) \equiv \left\{\begin{array}{llll} z^3 & & & \\ & +3z^2+3z & & \\ & & +3^2 z & \\ +2.3^3z^2+2.3^3z+2.3^3 & & & \end{array}\right\} \left\{\begin{array}{ll} z+1 & \\ & +2.3 \\ & +2.3^2 \\ & \end{array}\right\} (\text{mod } 3^4),$$

which is far enough for the purpose. No edge on either diagram passes through a node between its ends, so the second dissection completely separates the prime ideal factors of (3). In fact

$$(3) = \mathfrak{p}_3 \mathfrak{q}_3{}^2 \mathfrak{r}_3,$$

and from the $(z+1)$-expansion

$$\theta + 1 \equiv 0\ (\text{mod } \mathfrak{p}_3), \quad \theta + 7 \equiv 0\ (\text{mod } \mathfrak{p}_3{}^2) \text{ exactly}, \quad \theta + 16 \equiv 0\ (\text{mod } \mathfrak{p}_3{}^4).$$

The z-diagram shews that $\mathfrak{q}_3$, attached to Q_0Q_1, divides 3 twice and θ once, whereas $\mathfrak{r}_3$, attached to Q_1Q_2, divides 3 once and θ twice. After Chap. IX,

$$\begin{aligned} \chi_1^{(1)}(\theta) &= \theta + 1, \quad \chi_1^{(2)}(\theta) = \theta^2, \\ \chi_2^{(1)}(\theta) &= \theta + 7, \quad \chi_2^{(2)}(\theta) = \theta^3 + 3\theta^2 + 3\theta, \end{aligned}$$

whence, $\chi_2^{(1)}(\theta)\, \chi_2^{(2)}(\theta)$ being of rank 4,

$$\mathfrak{o} \equiv [1,\ \theta,\ \theta^2,\ \tfrac{1}{3}\theta^2(\theta+1)]\ (\text{bas } 3).$$

As a verification I note that $\alpha \equiv \frac{1}{3}\theta^2(\theta+1)$ satisfies

$$\alpha^4 - 164\alpha^3 - 3124\alpha^2 - 16240\alpha - 9936 = 0.$$

Since, further, $$\theta + 7 \equiv 0 \pmod{\mathfrak{p}_3^2},$$

$$\mathfrak{p}_3 \equiv [3,\ \theta+1,\ \theta^2+2,\ \tfrac{1}{3}\theta^2(\theta+7)]\ (\text{bas } 3).$$

Again from the expansion or diagram

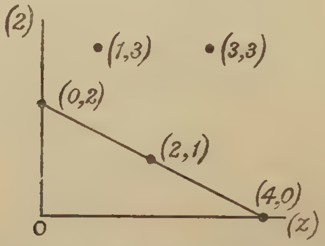
$$\theta^2 + 3\theta + 3 \equiv 0 \pmod{\mathfrak{q}_3^3}, \quad \theta^2 \equiv 0 \pmod{\mathfrak{r}_3^2},$$

so $$\mathfrak{q}_3 \equiv [3,\ \theta,\ \theta^2,\ \tfrac{1}{3}(\theta^2+3\theta+3)(\theta+1)]\ (\text{bas } 3),$$

and $$\mathfrak{r}_3 \equiv [3,\ \theta,\ \theta^2,\ \tfrac{1}{3}\theta^2(\theta+1)]\ (\text{bas } 3).$$

The only irreducible divisor of $a(z)$ (mod 2) is z, and the only marked nodes are those representing the terms

$$z^4,\ 2^3z^3,\ 3.2z^2,\ 3.2^3z,\ 27.2^2.$$



Since the one edge of the boundary passes through three nodes, the second dissection is insufficient to separate the prime ideal factors of (2). But since $z^4 + 3.2z^2 - 27.2^2$ is irreducible (modd $\{\sqrt{2},\ z\}^5$) the third dissection applies, and $(2) = \mathfrak{p}_2^2$, $\mathfrak{p}_2$ being a prime ideal of the second degree dividing θ once and (from the diagram)

$$\theta^4 + 3.2\theta^2 - 27.2^2$$

seven times. Evidently

$$\mathfrak{o} \equiv (1,\ \theta,\ \tfrac{1}{2}\theta^2,\ \tfrac{1}{2}\theta^3)\ (\text{bas } 2), \quad \mathfrak{p}_2 \equiv (2,\ \theta,\ \theta^2,\ \tfrac{1}{2}\theta^3)\ (\text{bas } 2).$$

Combining the partial bases of $\mathfrak{o}$ (bas 2) and (bas 3), which are the only ones,

$$\mathfrak{o} = [1,\ \theta,\ \tfrac{1}{2}\theta^2,\ \tfrac{1}{6}\theta^2(\theta+1)] = (1,\ \theta,\ \omega_2,\ \omega_3);$$

whence the discriminant of the field is

$$D = D(\theta) \div 2^4.3^2 = -2^6.3.11.13.401.$$

Since $$z^3 + 4z^2 \equiv \begin{cases} z^3 & \pmod 2, \\ z^3 + z^2 & \pmod 3, \end{cases}$$

$$\mathfrak{p}_2 = [2,\ \theta,\ \theta^2,\ \tfrac{1}{6}(\theta^3 + 4\theta^2)] = (2,\ \theta,\ 2\omega_2,\ \omega_3 + \omega_2).$$

Similarly $$\mathfrak{p}_3 = (3,\ \theta+1,\ \omega_2+1,\ \omega_3+1),$$

$$\mathfrak{q}_3 = (3,\ \theta,\ \omega_2,\ \omega_3+2), \quad \mathfrak{r}_3 = (3,\ \theta,\ \omega_2,\ \omega_3).$$

Returning to the prime ideal factors of (13), their bases are seen to be

$$\mathfrak{p}_{13} = (13,\ \theta + 9,\ \omega_2 + 5,\ \omega_3 + 4),$$
$$\mathfrak{q}_{13} = (13,\ \theta + 6,\ \omega_2 + 8,\ \omega_3 + 4),$$
$$\mathfrak{r}_{13} = (13,\ \theta + 7,\ \omega_2 + 8,\ \omega_3 + 10).$$

11 and 401, the remaining prime factors of the discriminant, are also each divisible by the square of a prime ideal. The irreducible divisors of $a(z)$ being

$$(z+2)^2 (z^2 + 10z + 6) \pmod{11},$$

and
$$(z + 316)^2 (z + 107)(z + 55) \pmod{401},$$

it immediately follows that

$$(11) = \mathfrak{p}_{11}{}^2 \mathfrak{q}_{11}, \quad (401) = \mathfrak{p}_{401}{}^2 \mathfrak{q}_{401} \mathfrak{r}_{401},$$

where
$$\mathfrak{p}_{11} = (11,\ \theta + 2,\ \omega_2 + 9,\ \omega_3 + 8),$$
$$\mathfrak{q}_{11} = (11,\ 11\theta,\ \omega_2 + 5\theta + 3,\ \omega_3 + 8\theta + 2),$$
$$\mathfrak{p}_{401} = (401,\ \theta + 316,\ \omega_2 + 197,\ \omega_3 + 167),$$
$$\mathfrak{q}_{401} = (401,\ \theta + 107,\ \omega_2 + 90,\ \omega_3 + 28),$$
$$\mathfrak{r}_{401} = (401,\ \theta + 55,\ \omega_2 + 292,\ \omega_3 + 358).$$

The prime ideal factors of any prime number p, other than 2, 3, 11, 13, 401 which divide the discriminant, are all different from each other and can be put down at once when the irreducible divisors of $a(z)$ (mod p) have been determined. When $p = 5$ the irreducible divisors are

$$z + 3, \quad z^3 + 4z^2 + 4z + 4,$$

so $(5) = \mathfrak{p}_5 \mathfrak{q}_5$ where, since

$$(z+3)(z-3) \equiv z^2 + 3 \,.\, 2 \pmod{5},$$
$$(z+3)(z^2 - 2z + 6) \equiv z^2 (z+1) + 3 \,.\, 6 \pmod{5},$$
$$z^3 + 4z^2 + 4z + 4 \equiv (z^3 + z^2) + 3z^2 + 4 \,.\, 6z + 4 \,.\, 6 \pmod{5},$$
$$\mathfrak{p}_5 = (5,\ \theta + 3,\ \omega_2 + 3,\ \omega_3 + 3),$$
$$\mathfrak{q}_5 = (5,\ 5\theta,\ 5\omega_2,\ \omega_3 + \omega_2 + 4\theta + 4).$$

Again

$$a(z) \equiv (z + 246)(z + 925)(z + 1248)(z + 1679) \pmod{2053},$$

so
$$(2053) = \mathfrak{p}_{2053} \mathfrak{q}_{2053} \mathfrak{r}_{2053} \mathfrak{s}_{2053}$$

with
$$\mathfrak{p}_{2053} = (2053,\ \theta + 246,\ \omega_2 = 537,\ \omega_3 + 1311),$$
$$\mathfrak{q}_{2053} = (2053,\ \theta + 925,\ \omega_2 = 238,\ \omega_3 + 604),$$
$$\mathfrak{r}_{2053} = (2053,\ \theta + 1248,\ \omega_2 = 1388,\ \omega_3 + 632),$$
$$\mathfrak{s}_{2053} = (2053,\ \theta + 1679,\ \omega_2 = 1917,\ \omega_3 + 1477).$$

When $p = 3121$ the irreducible divisors of $a(z)$ are

$$(z^2 + 369z + 2348)(z^2 + 2744z + 2568) \pmod{p},$$

whereas $a(z)$ is irreducible (mod 17). Accordingly

$$(17) \equiv (17,\ 17\theta,\ 17\omega_2,\ 17\omega_3)$$

is a prime ideal, while (3121) is the product of the prime ideals

$$(3121,\ 3121\theta,\ \omega_2 + 1745\theta + 1174,\ \omega_3 + 1687\theta + 2693),$$
$$(3121,\ 3121\theta,\ \omega_2 + 1372\theta + 1284,\ \omega_3 + 1645\theta + 2613).$$

Finally $(41) = \mathfrak{p}_{41}\mathfrak{q}_{41}\mathfrak{r}_{41}$ where, since

$$a(z) \equiv (z^2 + 32z + 7)(z + 13)(z + 29) \pmod{41},$$
$$\mathfrak{p}_{41} = (41,\ 41\theta,\ \omega_2 + 16\theta + 24,\ \omega_3 + 34\theta + 39),$$
$$\mathfrak{q}_{41} = (41,\ \theta + 13,\ \omega_2 + 18,\ \omega_3 + 10),$$
$$\mathfrak{r}_{41} = (41,\ \theta + 29,\ \omega_2 + 10,\ \omega_3 + 16).$$

(It is no part of the purpose of this investigation to ascertain whether the prime ideals whose bases have been obtained are principal ideals or not.)

The only partitions of 4 as a sum of positive integers are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Among the above examples there is one prime number (17, 5, 3121, 41, 2053) breaking up into prime ideal factors whose degrees are equal to the members of each of these partitions. If $[\theta]$ is a field of degree n and

$$a + b + c + \dots + l$$

is a partition of n as a sum of positive integers, investigation is still required to decide whether there exist prime numbers whose prime ideal factors are of degrees $a, b, c, \dots l$. When the group of the field is symmetric it seems possible that an unlimited number of primes corresponds to each partition, but this remains to be proved.

61. II. $$a(\theta) \equiv \theta^9 + 539 \equiv \theta^9 + 7^2 . 11 = 0:$$
$$D(\theta) = N(9\theta^8) = 3^{18} . 7^{16} . 11^8.$$

The only irreducible divisor of $a(z)$ (mod 11) being z, the diagram only contains one edge—joining (9, 0) to (0, 1). It follows that

$$\mathfrak{o} \equiv (1,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5,\ \theta^6,\ \theta^7,\ \theta^8) \text{ (bas 11)},$$

and that $(11) = \mathfrak{p}_{11}{}^9$ where

$$\mathfrak{p}_{11} \equiv (11,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5,\ \theta^6,\ \theta^7,\ \theta^8) \text{ (bas 11)}.$$

Again z is the only irreducible divisor of $a(z)$ (mod 7) and the one edge of the diagram joins (9, 0) to (0, 2). Since this edge includes no node between its ends, but passes through $(4\tfrac{1}{2}, 1)$, the number $\tfrac{1}{7}\theta^5$ is integral and

$$\mathfrak{o} \equiv (1,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \tfrac{1}{7}\theta^5,\ \tfrac{1}{7}\theta^6,\ \tfrac{1}{7}\theta^7,\ \tfrac{1}{7}\theta^8)\ (\text{bas } 7),$$

whereas $(7) = \mathfrak{p}_7^{\,9}$ with

$$\mathfrak{p}_7 \equiv (7,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \tfrac{1}{7}\theta^5,\ \tfrac{1}{7}\theta^6,\ \tfrac{1}{7}\theta^7,\ \tfrac{1}{7}\theta^8)\ (\text{bas } 7).$$

Passing to the remaining factor 3 of $D(\theta)$,

$$a(z) \equiv (z-1)^9 \pmod{3},$$

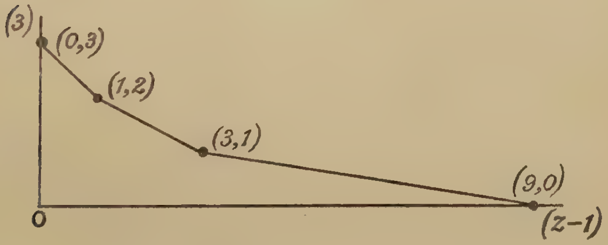
and the expansion of Chap. V can be taken to be

$$\begin{aligned} z^9 + 539 = {} & (z-1)^9 + 3^2 . (z-1)^8 + 4 . 3^2 . (z-1)^7 \\ & + 28 . 3\, (z-1)^6 + 14 . 3^2 . (z-1)^5 + 14 . 3^2 . (z-1)^4 + 28 . 3 . (z-1)^3 \\ & + 4 . 3^2 . (z-1)^2 + 3^2 . (z-1) + 20 . 3^3, \end{aligned}$$

giving a three-edged diagram which shews that

$$(3) = \mathfrak{p}_3^{\,6}\,\mathfrak{q}_3^{\,2}\,\mathfrak{r}_3,$$

$$\theta - 1 \equiv 0\ (\text{mod } \mathfrak{p}_3\mathfrak{q}_3\mathfrak{r}_3)\ \text{exactly.}$$



By the usual rule the stem of $\mathfrak{o}$ (bas 3) is

$$1,\ \frac{(\theta-1)^6}{3},\ \frac{(\theta-1)^8 + 28 . 3\,(\theta-1)^5 + 28 . 3\,(\theta-1)^2}{9},$$

and since
$$(z-1)^6 \equiv z^6 + z^3 + 1 \pmod{3},$$

$$(z-1)^8 + 84\,(z-1)^5 + 84\,(z-1)^2 \equiv z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 \pmod{9},$$

it can equally well be taken to be

$$1,\ \frac{\theta^6 + \theta^3 + 1}{3},\ \frac{\theta^8 + \theta^7 + \theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1}{9}.$$

To form stems for the prime ideal factors of 3 we note (from the diagram) that

$$(\theta-1)^9+28\,.\,3\,(\theta-1)^3\equiv 0 \pmod{\mathfrak{p}_3^{12}},$$

i.e.

$$(\theta-1)^6+28\,.\,3\equiv 0 \pmod{\mathfrak{p}_3^{9}},$$

or $\frac{1}{3}(\theta^6+\theta^3+1)+(\theta-1)(-2\theta^4+3\theta^3-4\theta^2+\theta-1)+27\equiv 0 \pmod{\mathfrak{p}_3^{3}}$

and

$$\tfrac{1}{3}(\theta^6+\theta^3+1)\equiv 0 \pmod{\mathfrak{p}_3}.$$

Similarly $\frac{1}{3}(\theta^6+\theta^3+1)\equiv 1 \pmod{\mathfrak{q}_3}$ and $\pmod{\mathfrak{r}_3}$,

$$\tfrac{1}{9}(\theta^8+\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1)\equiv 0 \pmod{\mathfrak{p}_3\mathfrak{q}_3},$$

$$\tfrac{1}{9}(\theta^8+\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1)\equiv 1 \pmod{\mathfrak{r}_3}.$$

Accordingly the respective stems of $\mathfrak{p}_3$, $\mathfrak{q}_3$, $\mathfrak{r}_3$ (bas 3) are

$3,\ \theta-1,\ \frac{1}{3}(\theta^6+\theta^3+1),\quad \frac{1}{9}(\theta^8+\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1);$

$3,\ \theta-1,\ \frac{1}{3}(\theta^6+\theta^3+1)-1,\ \frac{1}{9}(\theta^8+\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1);$

$3,\ \theta-1,\ \frac{1}{3}(\theta^6+\theta^3+1)-1,\ \frac{1}{9}(\theta^8+\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1)-1.$

Combining the partial bases of $\mathfrak{o}$ (bas 3) and (bas 7),

$$\mathfrak{o}=(1,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \omega_5,\ \omega_6,\ \theta\omega_6,\ \omega_8),$$

where

$$7\omega_5=\theta^5,\quad 21\omega_6=\theta^6+7\theta^3+7,$$

and

$$63\omega_8=\theta^8+\theta^7+\theta^6+\theta^5+28\,(\theta^4+\theta^3+\theta^2+\theta+1).$$

Also

$$D=D(\theta)\div 7^8\,.\,3^8=3^{10}\,.\,7^8\,.\,11^8.$$

The bases of the prime ideal factors of 11, 7, 3 now readily follow:

$\mathfrak{p}_{11}=(11,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \omega_5,\ \omega_6+7,\ \theta\omega_6,\ \omega_8+2),$

$\mathfrak{p}_7=(7,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \omega_5,\ \omega_6+2,\ \theta\omega_6,\ \omega_8+5),$

$\mathfrak{p}_3=(3,\ \theta-1,\ \theta^2-1,\ \theta^3-1,\ \theta^4-1,\ \omega_5-1,\ \omega_6-1,\ \theta\omega_6-1,\ \omega_8),$

$\mathfrak{q}_3=(3,\ \theta-1,\ \theta^2-1,\ \theta^3-1,\ \theta^4-1,\ \omega_5-1,\ \omega_6+1,\ \theta\omega_6+1,\ \omega_8),$

$\mathfrak{r}_3=(3,\ \theta-1,\ \theta^2-1,\ \theta^3-1,\ \theta^4-1,\ \omega_5-1,\ \omega_6+1,\ \theta\omega_6+1,\ \omega_8-1).$

To determine the prime ideal factors of any other prime number p it is necessary to find the irreducible divisors of $z^9+539 \pmod{p}$. For example,

$$z^9+539\equiv(z+1)(z^2+z+1)(z^6+z^3+1) \pmod{2},$$

so (2) is the product of the three prime ideals

$(2,\ \theta+1,\ \theta^2+1,\ \theta^3+1,\ \theta^4+1,\ \omega_5+1,\ \omega_6+1,\ \theta\omega_6+1,\ \omega_8),$

$(2,\ 2\theta,\ \theta^2+\theta+1,\ \theta^3+1,\ \theta^4+\theta,\ \omega_5+\theta+1,\ \omega_6+1,\ \theta\omega_6+\theta,\ \omega_8+\theta+1),$

$(2,\ 2\theta,\ 2\theta^2,\ 2\theta^3,\ 2\theta^4,\ 2\omega_5,\ \omega_6,\ \theta\omega_6,\ \omega_8+\theta^4+\theta^3+\theta^2+\theta+1).$

The complex integers of the field defined by $\theta^n-a=0$, of which this is a particular case, are discussed in some detail in Chap. XV below.

CHAPTER XII

RELATIVE FIELDS

62. If $\alpha_1, \alpha_2, \dots \alpha_n$ lie in a number-field $[\alpha]$ of degree m, and if the polynomial

$$f(\theta, \alpha) \equiv \theta^n + \alpha_1 \theta^{n-1} + \dots + \alpha_n$$

is irreducible in $[\alpha]$, θ is an algebraic number of degree mn, satisfying the rational irreducible equation

$$\Pi_{(\alpha)} f(\theta, \alpha) = 0.$$

The field $[\theta]$, or $[\theta, \alpha]$, is here to be discussed as a relative field (or super-field) of $[\alpha]$. In order that θ may be integral it is necessary and sufficient for $\alpha_1, \dots \alpha_n$ to be integral in the sub-field $[\alpha]$. The integral basis of $[\alpha]$ is $\mathfrak{o} \equiv (\omega_0, \omega_1, \dots \omega_{m-1})$.

63. The general member of $[\theta]$ is uniquely expressible as

$$y(\theta) \equiv y_1 \theta^{n-1} + y_2 \theta^{n-2} + \dots + y_n$$

where the y's belong to $[\alpha]$. As usual in this theory I put

$$N_\alpha(\theta) = (-)^n \alpha_n, \quad D_\alpha(\theta) = \prod_{r,s} (\theta_r - \theta_s)^2,$$

these being respectively the relative-norm and relative-discriminant of θ in $[\alpha]$. When $y(\theta)$ is integral it follows as in Chap. III that $y_i = \mu_i / D_\alpha(\theta)$, μ_i being an integer of $[\alpha]$. Hence every integer of $[\theta]$ is expressible in the form

$$(\mu_1 \theta^{n-1} + \mu_2 \theta^{n-2} + \dots + \mu_n) / D_\alpha(\theta),$$

but of course all numbers of this type are not integral.

Those integers of $[\theta]$ which belong to $[\alpha]$ are also integers therein, and form the integral basis $\mathfrak{o}$ of the sub-field.

Again if $\beta_1, \beta_2, \dots,$ integers of $[\alpha]$, belong to an ideal $\mathfrak{B}$ of $[\theta]$, so also does

$$\omega_r \beta_1 + \omega_s \beta_2 + \dots.$$

The aggregate of such integers β_i is therefore an ideal $\mathfrak{b}$ of $[\alpha]$.

64. At least one integer of rank s, viz.: $\theta^s N(\mathfrak{A})$, belongs to an ideal $\mathfrak{A}$. If

$$\Omega_s \equiv \frac{\lambda_0 \theta^s + \lambda_1 \theta^{s-1} + \dots + \lambda_s}{D_\alpha(\theta)}$$

belongs to $\mathfrak{A}$, so also do the integers

$$\omega_0 \Omega_s, \ \omega_1 \Omega_s, \ \dots \ \omega_{n-1} \Omega_s.$$

Hence the aggregate of the integers λ_0 is an ideal $\mathfrak{a}_s$ of $[\alpha]$.

65. Since $\theta\Omega_{s-1}$ belongs to $\mathfrak{A}$, $\mathfrak{a}_s$ divides $\mathfrak{a}_{s-1}$.

Put
$$\mathfrak{a}_{s-1}=\mathfrak{k}_s\mathfrak{a}_s,$$
$$\mathfrak{a}_0=\mathfrak{a}D(\theta)=\mathfrak{k}_1\mathfrak{a}_1=\mathfrak{k}_1\mathfrak{k}_2\dots\mathfrak{k}_s\mathfrak{a}_s.$$

66. Next considering integers of the first rank contained in $\mathfrak{A}$, let κ be a member of $\mathfrak{k}_1$ so chosen that $(\kappa,\ \mathfrak{a}D_\alpha(\theta))=\mathfrak{k}_1$. Then since
$$\frac{\kappa\lambda_1}{D_\alpha(\theta)}=\kappa\frac{\lambda_0\theta+\lambda_1}{D_\alpha(\theta)}-\frac{\kappa\lambda_0}{D_\alpha(\theta)}\theta,$$
and
$$\kappa\frac{\lambda_0\theta+\lambda_1}{D\ (\theta)}\equiv 0\ (\text{mod}\ \mathfrak{A}\mathfrak{k}_1),\quad \frac{\kappa\lambda_0}{D_\alpha(\theta)}\equiv 0\ (\text{mod}\ \mathfrak{a}),$$
it follows that
$$\kappa\lambda_1/D_\alpha(\theta)\equiv 0\ (\text{mod}\ \mathfrak{A}).$$
Since all members of $\mathfrak{A}$ which belong to $[\alpha]$ are also $\equiv 0\ (\text{mod}\ \mathfrak{a})$ the last congruence involves
$$\kappa\lambda_1\equiv 0\ (\text{mod}\ \mathfrak{a}D(\theta)),\ \text{i.e.}\ (\text{mod}\ \mathfrak{k}_1\mathfrak{a}_1),$$
whence $\lambda_1\equiv 0\ (\text{mod}\ \mathfrak{a}_1)$ whatever the value of λ_0 in $\mathfrak{a}_1$. Choosing λ_0 so that $(\lambda_0,\ \mathfrak{a}D(\theta))=\mathfrak{a}_1$ the congruence
$$\lambda_0\beta_{11}\equiv\lambda_1\ (\text{mod}\ \mathfrak{a}D(\theta))$$
determines β_{11} uniquely $(\text{mod}\ \mathfrak{k}_1)$. It is now readily seen that if μ_0 is any member of $\mathfrak{a}_1$ the integer $\mu_0(\theta+\beta_{11})/D_\alpha(\theta)$ belongs to $\mathfrak{A}$. The aggregate of first rank integers contained in $\mathfrak{A}$ is thus
$$\mathfrak{a},\ \frac{\mathfrak{a}_1\psi_1(\theta)}{D_\alpha(\theta)}\ \text{with}\ \psi_1(\theta)=\theta+\beta_{11}.$$

67. Repeating the argument of § 66 the complete basis of the relative ideal $\mathfrak{A}$ is
$$\mathfrak{a},\ \frac{\mathfrak{a}_1\Psi_1(\theta)}{D_\alpha(\theta)},\ \dots\ \frac{\mathfrak{a}_{n-1}\Psi_{n-1}(\theta)}{D_\alpha(\theta)}\quad\dots\dots\dots\dots(29)$$
with
$$\Psi_s(\theta)=\theta^s+\beta_{s1}\theta^{s-1}+\dots+\beta_{ss}.$$

68. The integral basis of $[\theta]$ is the ideal
$$\mathfrak{O}\equiv\left(\mathfrak{o},\ \frac{\mathfrak{d}_1\psi_1(\theta)}{D_\alpha(\theta)},\ \dots\ \frac{\mathfrak{d}_{n-1}\psi_{n-1}(\theta)}{D_\alpha(\theta)}\right)\dots\dots\dots(30),$$
where
$$\psi_s(\theta)=\theta^s+a_{s1}\theta^{s-1}+\dots+a_{ss}$$
and each of the ideals $\mathfrak{d}_0\equiv\mathfrak{o}D_\alpha(\theta),\ \mathfrak{d}_1,\ \dots\ \mathfrak{d}_{n-1}$ is a factor of all those on its left.

When all the ideals $\mathfrak{d}_1,\ \dots\ \mathfrak{d}_{n-1}$ are principal and respectively equal to $(\delta_1),\ \dots\ (\delta_{n-1})$,
$$\mathfrak{O}=\mathfrak{o}\left(1,\ \frac{\delta_1\Psi_1(\theta)}{D_\alpha(\theta)},\ \dots\ \frac{\delta_{n-1}\Psi_{n-1}(\theta)}{D_\alpha(\theta)}\right).$$
For the integral basis of the relative field to be of this latter special type it is clearly sufficient (but not necessary) that the class-number of $[\alpha]$ should be unity. It should, however, be noted that as yet there are

only three types of number-field in which the class-number can infallibly be determined by a finite number of arithmetical operations. The solution of this problem for a real or imaginary quadratic field (in the notation of quadratic forms) is due to Gauss: that for a cubic field of negative discriminant has been given by the present writer. (A quartic field whose conjugates are all imaginary can be handled on the same lines as the last mentioned investigation.)

Results equivalent in substance to those of this paragraph are also due to G. E. Wahlin. Kronecker in *Grundzüge* (§§ 6, 7) also gives general theorems on the integral basis of any arithmetical field.

69. Since every member of $\mathfrak{A}$ is included in $\mathfrak{O}$, $\mathfrak{d}_s$ (of § 68) divides $\mathfrak{a}_s$ (of § 67). Putting $\mathfrak{a}_s = \mathfrak{n}_s \mathfrak{d}_s$ the ideal

$$\mathfrak{n}_0 \mathfrak{n}_1 \dots \mathfrak{n}_{n-1} \equiv N_\alpha(\mathfrak{A})$$

is the relative-norm of $\mathfrak{A}$.

When $\mathfrak{A}$ is a prime ideal $\mathfrak{P}$, $\mathfrak{n}_0 = \mathfrak{p}$, a prime ideal of $[\alpha]$, and each of the ideals $\mathfrak{n}_s = \mathfrak{p}$ or $\mathfrak{o}$. Hence

$$N_\alpha(\mathfrak{P}) = \mathfrak{p}^t, \quad 0 < t \leqslant n.$$

The ideal $\mathfrak{I}$ belongs to $[\alpha]$ when it is the greatest common measure in $[\theta]$ of a set of integers $\gamma_1, \gamma_2, \dots$ of the sub-field. If

$$\mathfrak{j} \equiv (L_1, L_2, \dots L_m)$$

is a basis of the ideal $(\gamma_1, \gamma_2, \dots)$ in $[\alpha]$ it is readily seen that

$$\left(\mathfrak{j}, \frac{\mathfrak{j}\mathfrak{d}_1 \psi_1(\theta)}{D_\alpha(\theta)}, \dots \frac{\mathfrak{j}\mathfrak{d}_{n-1}\psi_{n-1}(\theta)}{D_\alpha(\theta)}\right)$$

is a basis of $\mathfrak{I}$ which $= \mathfrak{j}\mathfrak{O}$ and that

$$N_\alpha(\mathfrak{I}) = \mathfrak{j}^n.$$

70. If $\omega, \delta_1, \delta_2, \dots \delta_{n-1}$ are typical members of $\mathfrak{o}, \mathfrak{d}_1, \mathfrak{d}_2, \dots \mathfrak{d}_{n-1}$ respectively, and $\theta^{(1)}, \theta^{(2)}, \dots \theta^{(n)}$ the conjugate roots of $f(\theta, \alpha) = 0$, the square of the determinant

$$\begin{vmatrix} \omega, & \dfrac{\delta_1 \psi_1(\theta^{(1)})}{D_\alpha(\theta)}, & \dfrac{\delta_2 \psi_2(\theta^{(1)})}{D_\alpha(\theta)}, & \dots & \dfrac{\delta_{n-1}\psi_{n-1}(\theta^{(1)})}{D_\alpha(\theta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega, & \dfrac{\delta_1 \psi_1(\theta^{(n)})}{D_\alpha(\theta)}, & \dfrac{\delta_2 \psi_2(\theta^{(n)})}{D_\alpha(\theta)}, & \dots & \dfrac{\delta_{n-1}\psi_{n-1}(\theta^{(n)})}{D_\alpha(\theta)} \end{vmatrix}$$

belongs to $\mathfrak{o}$. The greatest common divisor in $\mathfrak{o}$ of all such squared determinants is the ideal

$$\mathfrak{D} \equiv \frac{\mathfrak{d}_1{}^2 \mathfrak{d}_2{}^2 \dots \mathfrak{d}^2{}_{n-1}}{D_\alpha(\theta)^{2n-3}}$$

which is the *relative discriminant* of $[\theta]$ with respect to $[\alpha]$. On bringing this notation into line with that used by Hilbert (*Zahlbericht*, § 14) it is readily seen that $\mathfrak{D}$ is identical with the relative-discriminant there defined.

If d is the discriminant of $[\alpha]$ and D that of $[\theta]$ the known relation

$$D = d^n N(\mathfrak{D})$$

(*Zahlbericht*, § 15) can be established without difficulty from the above definition of $\mathfrak{D}$. For convenience I take $m = 3$, $n = 2$, but the argument is general.

The integral basis of $[\alpha]$ is $(\omega_0, \omega_1, \omega_2)$ and those of $[\alpha']$, $[\alpha'']$ are $(\omega_0', \omega_1', \omega_2')$, $(\omega_0'', \omega_1'', \omega_2'')$. Taking θ_1, θ_2 to be the roots of

$$\theta^2 + \alpha_1 \theta + \alpha_2 = 0,$$

and similarly for θ_1', θ_2' and θ_1'', θ_2'' the integral basis of $[\theta_1, \alpha]$ can be taken to be

$$\begin{aligned}[\omega_0,\ &\omega_1,\ \omega_2;\\ &\delta^{-1}\{(l_0\omega_0 + l_1\omega_1 + l_2\omega_2)\,\theta_1 + m_0\omega_0 + m_1\omega_1 + m_2\omega_2\},\\ &\delta^{-1}\{(l_0'\omega_0 + l_1'\omega_1 + l_2'\omega_2)\,\theta_1 + m_0'\omega_0 + m_1'\omega_1 + m_2'\omega_2\},\\ &\delta^{-1}\{(l_0''\omega_0 + l_1''\omega_1 + l_2''\omega_2)\,\theta_1 + m_0''\omega_0 + m_1''\omega_1 + m_2''\omega_2\}]\end{aligned}$$

where $\delta = ND_\alpha(\theta)$. $\sqrt{D}$ is a six-rowed determinant of which the above six numbers form the first row, the other five being obtained by changing $(\omega_0, \omega_1, \omega_2, \theta_1)$ into

$$(\omega_0, \omega_1, \omega_2, \theta_2),\quad (\omega_0', \omega_1', \omega_2', \theta_1'),\quad (\omega_0', \omega_1', \omega_2', \theta_2'),$$
$$(\omega_0'', \omega_1'', \omega_2'', \theta_1'') \text{ and } (\omega_0'', \omega_1'', \omega_2'', \theta_2'')$$

respectively. Hence

$$\sqrt{D} = \frac{1}{\delta^3}\begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ m_0 & m_1 & m_2 & l_0 & l_1 & l_2 \\ m_0' & m_1' & m_2' & l_0' & l_1' & l_2' \\ m_0'' & m_1'' & m_2'' & l_0'' & l_1'' & l_2'' \end{vmatrix} \times \begin{vmatrix} \omega_0 & \omega_1 & \omega_2 & \omega_0\theta_1 & \omega_1\theta_1 & \omega_2\theta_1 \\ \omega_0 & \omega_1 & \omega_2 & \omega_0\theta_2 & \omega_1\theta_2 & \omega_2\theta_2 \\ \omega_0' & \omega_1' & \omega_2' & \omega_0'\theta_1' & \omega_1'\theta_1' & \omega_2'\theta_1' \\ \omega_0' & \omega_1' & \omega_2' & \omega_0'\theta_2' & \omega_1'\theta_2' & \omega_2'\theta_2' \\ \omega_0'' & \omega_1'' & \omega_2'' & \omega_0''\theta_1'' & \omega_1''\theta_1'' & \omega_2''\theta_1'' \\ \omega_0'' & \omega_1'' & \omega_2'' & \omega_0''\theta_2'' & \omega_1''\theta_2'' & \omega_2''\theta_2'' \end{vmatrix}$$

$$= \frac{\begin{vmatrix} l_0 & l_1 & l_2 \\ l_0' & l_1' & l_2' \\ l_0'' & l_1'' & l_2'' \end{vmatrix}}{\delta^3} \begin{vmatrix} 1 & \theta_1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \theta_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \theta_1' & \cdot & \cdot \\ \cdot & \cdot & 1 & \theta_2' & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \theta_3 \\ \cdot & \cdot & \cdot & \cdot & 1 & \theta_3' \end{vmatrix} \begin{vmatrix} \omega_0 & \omega_1 & \omega_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega_0 & \omega_1 & \omega_2 \\ \omega_0' & \omega_1' & \omega_2' & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega_0' & \omega_1' & \omega_2' \\ \omega_0'' & \omega_1'' & \omega_2'' & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega_0'' & \omega_1'' & \omega_2'' \end{vmatrix}$$

$$= \delta^{-3}(l_0 l_1' l_2'')\,(\theta_1 - \theta_2)\,(\theta_1' - \theta_2')\,(\theta_1'' - \theta_2'')\,d,$$

and $$D = \delta^{-6}(l_0 l_1' l_2'')^2\,D_\alpha(\theta)\,D_\alpha'(\theta')\,D_\alpha''(\theta'')\,d^2.$$

Also

$$\mathfrak{D} = \delta^{-2} D_\alpha(\theta)\,(l_0\omega_0 + l_1\omega_1 + l_2\omega_2,\ l_0'\omega_0 + l_1'\omega_1 + l_2'\omega_2,\ l_0''\omega_0 + l_1''\omega_1 + l_2''\omega_2)^2$$

and $$N(\mathfrak{D}) = \delta^{-6} D_\alpha(\theta)\, D_\alpha'(\theta')\, D_\alpha''(\theta'')\,(l_0 l_1' l_2'')^2.$$

Hence, immediately, $D = d^2 N(\mathfrak{D})$.

71. Writing $\mathfrak{o} D_\alpha(\theta) = \mathfrak{d}_i \mathfrak{d}_i'$ the basis (30) of $\mathfrak{O}$ is

$$\left(\mathfrak{o},\ \frac{\mathfrak{o}\psi_1(\theta)}{\mathfrak{d}_1'},\ \frac{\mathfrak{o}\psi_2(\theta)}{\mathfrak{d}_2'},\ \dots\ \frac{\mathfrak{o}\psi_{n-1}(\theta)}{\mathfrak{d}'_{n-1}}\right),$$

and each of the ideals

$$\mathfrak{d}_1',\ \mathfrak{d}_2',\ \dots\ \mathfrak{d}'_{n-1},\ \mathfrak{o} D_\alpha(\theta)$$

is a factor of all those on its right.

In the field $[\theta, \alpha]$ a prime ideal $\mathfrak{p}$ of $[\alpha]$ plays precisely the same part as a prime number p plays in $[\alpha]$. Just as the prime factors of $D(\theta)$ were separated in Chap. IV, so can the prime ideal factors of $\mathfrak{o} D_\alpha(\theta)$ be separated here. The integral basis $\mathfrak{O}$ is thus split into a number of partial bases—one for each distinct prime ideal factor of $\mathfrak{o} D_\alpha(\theta)$, and each partial basis possesses a stem. Again, the polynomial $f(z, \alpha)$ is congruent to an unique product of irreducible divisors (mod $\mathfrak{p}$) and its unique expansion (mod $\mathfrak{p}^{d+1}$) is valid too, just as in Chap. V. In fact the whole theory of these divisors is true practically verbatim when the coefficients $a_1, \dots a_n$ belong to $\mathfrak{o}$ and the modulus is a power of a prime ideal. Further, the properties of the prime ideals and of ϕ's and χ's proved in Chaps. VI–VIII are of a quite general arithmetical character, and the methods given in Chaps. IX, X for constructing them can all be applied to the relative-field. The whole investigation, in fact, could well have been given in ideal notation from the beginning.

The examples given in the next chapter will sufficiently indicate the methods to be adopted in dealing with a numerical case.

CHAPTER XIII

FURTHER NUMERICAL ILLUSTRATIONS

72. III. $\alpha = \frac{1}{2}(1 + \sqrt{57}),\ \ \alpha^2 - \alpha - 14 = 0,$

$\mathfrak{o} = (1, \alpha)\quad d = d(\alpha) = 57 = 3\,.\,19.$

$(3) = (3, 1 + \alpha)^2 = \mathfrak{p}_3^2,\ \ (19) = (19, 9 + \alpha)^2 = \mathfrak{p}_{19}^2,$

$(2) = (2, \alpha)(2, 1 + \alpha) = \mathfrak{p}_2 \mathfrak{q}_2,$

$(7) = (7, \alpha)(7, 6 + \alpha) = \mathfrak{p}_7 \mathfrak{q}_7.$

A relative-quadratic field is defined by

$$\beta(\theta) \equiv \theta^2 - 10 - \alpha = 0$$

and $$\mathfrak{o}D_\alpha(\theta) = 4(10+\alpha)\,\mathfrak{o} = (2,\ \alpha)^7\,(2,\ 1+\alpha)^2\,(3,\ 1+\alpha).$$

Since $$\beta(z) = (z-1)^2 + 2(z-1) - 9 - \alpha$$

and $$(9+\alpha) = \mathfrak{q}_2^{\,2}\,\mathfrak{p}_{19},$$

$$\beta(z) \equiv (z-1)^2 + 2(z-1) + 4 \ (\text{modd}\ \{z-1,\ \mathfrak{q}_2\}^3)$$

which is irreducible. Noting that the one edge of the $\mathfrak{q}_2$ diagram passes through three marked nodes, (2, 0), (1, 1), (0, 2), it is evident that

$$\mathfrak{O} \equiv \left(\mathfrak{o},\ \frac{\mathfrak{o}(\theta-1)}{\mathfrak{q}_2}\right) \equiv \left(\mathfrak{o},\ \frac{\mathfrak{p}_2(\theta-1)}{2}\right) (\text{bas } \mathfrak{q}_2)$$

and that $$\mathfrak{O}\mathfrak{q}_2 = \mathfrak{Q}_2 \equiv (\mathfrak{q}_2,\ \mathfrak{o}(\theta-1))\ (\text{bas } \mathfrak{q}_2)$$

is a prime ideal of the second degree.

Next since $10+\alpha \equiv 0 \pmod{\mathfrak{p}_2^{\,5}}$ exactly, the $\mathfrak{p}_2$-diagram has one edge joining (2, 0), (0, 5) and

$$\mathfrak{p}_2\mathfrak{O} = \mathfrak{P}_2^{\,2}$$

where $$\mathfrak{O} \equiv \left(\mathfrak{o},\ \frac{\mathfrak{o}\theta}{\mathfrak{p}_2^{\,2}}\right) \equiv \left(\mathfrak{o},\ \frac{\mathfrak{q}_2^{\,2}\theta}{4}\right) (\text{bas } \mathfrak{p}_2),$$

$$\mathfrak{P}_2 \equiv \left(\mathfrak{p}_2,\ \frac{\mathfrak{o}\theta}{\mathfrak{p}_2^{\,2}}\right) \equiv \left(\mathfrak{p}_2,\ \frac{\mathfrak{q}_2^{\,2}\theta}{4}\right) (\text{bas } \mathfrak{p}_2).$$

Noting that $$z + 2 + \alpha \equiv \begin{cases} z \pmod{\mathfrak{p}_2^{\,2}}, \\ z - 1 \pmod{\mathfrak{q}_2}, \end{cases}$$

$$\mathfrak{O} = \left(\mathfrak{o},\ \frac{\mathfrak{q}_2(\theta+2+\alpha)}{4}\right) = [1,\ \alpha;\ \tfrac{1}{4}(2,\ 1+\alpha)(\theta+2+\alpha)],$$

also $$\mathfrak{D} = (2,\ \alpha)^3\,(3,\ 1+\alpha),$$

and $$D = d^2 N(\mathfrak{D}) = 2^3 \cdot 3^3 \cdot 19^2.$$

The bases of the prime ideal factors of $2\mathfrak{O}\ (= \mathfrak{P}_2^{\,2}\mathfrak{Q}_2)$ are

$$\mathfrak{P}_2 = [\mathfrak{p}_2,\ \tfrac{1}{4}\mathfrak{q}_2(\theta+2+\alpha)] = [2,\ \alpha;\ \tfrac{1}{4}(2,\ 1+\alpha)(\theta+2+\alpha)],$$
$$\mathfrak{Q}_2 = [\mathfrak{q}_2,\ \tfrac{1}{4}\mathfrak{q}_2^{\,2}(\theta+2+\alpha)] = [2,\ 1+\alpha;\ \tfrac{1}{4}(4,\ 1+\alpha)(\theta+2+\alpha)].$$

To factorise any other prime ideal $\mathfrak{p}$ of $[\alpha]$ it is only necessary to find the irreducible divisors of $\beta(z) \pmod{\mathfrak{p}}$. Thus

$$\beta(z) \equiv (z+1)(z-1) \pmod{\mathfrak{p}_{19}}$$

so, since $$10 + \alpha \equiv \begin{cases} 1 \pmod{\mathfrak{p}_{19}}, \\ \pm(2+\alpha) \pmod{\mathfrak{p}_2^{\,2}\mathfrak{q}_2}, \end{cases}$$

$\mathfrak{p}_{19}$ is the product of the two prime ideals

$$[19,\ 9+\alpha;\ \tfrac{1}{4}(2,\ 1+\alpha)(\theta-10-\alpha)],$$
$$[19,\ 9+\alpha;\ \tfrac{1}{4}(2,\ 1+\alpha)(\theta+10+\alpha)].$$

$(3, 1+\alpha)$, the remaining factor of $\mathfrak{D}$, is the square of

$$[3, 1+\alpha;\ \tfrac{1}{4}(2, 1+\alpha)(\theta+2-\alpha)].$$

$$\mathfrak{O}(7, \alpha) \equiv [7, \alpha;\ \tfrac{1}{4}(14, 7+\alpha)(\theta+2+\alpha)]$$

is a prime ideal of the second degree and an ideal of $[\alpha]$ (*cf.* § 69), but $(7, 6+\alpha)\,\mathfrak{O}$ is the product of two prime ideals of the first degree, viz.:

$$[7, 6+\alpha;\ \tfrac{1}{4}(2, 1+\alpha)(\theta-22-\alpha)]$$

and

$$[7, 6+\alpha;\ \tfrac{1}{4}(2, 1+\alpha)(\theta+22+\alpha)].$$

$(97, 97\alpha)$ and $(79, 79\alpha)$ are prime ideals in $[\alpha]$. Since $\beta(z)$ is irreducible (mod $79\mathfrak{o}$) but

$$\equiv (z+29+10\alpha)(z-29-10\alpha) \pmod{97\mathfrak{o}},$$

$79\mathfrak{O}$ is prime in $[\theta]$ while $97\mathfrak{O}$ is the product of

$$[97, 97\alpha;\ \tfrac{1}{4}(2, 1+\alpha)(\theta+126+107\alpha)]$$

and

$$[97, 97\alpha;\ \tfrac{1}{4}(2, 1+\alpha)(\theta-126-107\alpha)].$$

A long and classical memoir by Hilbert gives a detailed discussion of the fields defined by

$$\theta^2 = a + b\sqrt{m}$$

of which this affords a particular example.

73. IV. The quadratic field $[\sqrt{-23}]$ is defined by

$$a(\alpha) \equiv \alpha^2 - \alpha + 6 = 0;$$

its integral basis is $\mathfrak{o}_\alpha \equiv (1, \alpha)$

and discriminant $D_\alpha = D(\alpha) = -23.$

In $[\alpha]$, $(23) = (23, \alpha+11)^2 = \mathfrak{b}_{23}{}^2,$

$(47) = (47, \alpha+13)(47, \alpha+33) = \mathfrak{b}_{47}\mathfrak{c}_{47},$

$(53) = (53, 53\alpha) = \mathfrak{b}_{53}$, a prime ideal, and

$(59) = (59, \alpha+26)(59, \alpha+32) = \mathfrak{b}_{59}\mathfrak{c}_{59}.$

Calling $\theta, \theta', \theta''$ the conjugate roots of

$$b(\theta) \equiv \theta^3 - \theta - 1 = 0$$

the integral basis of $[\theta]$ is $\mathfrak{o}_\theta \equiv (1, \theta, \theta^2)$ and its discriminant

$$D_\theta = D(\theta) = -23.$$

In $[\theta]$, $(23) = (23, \theta+13, \theta^2+15)^2(23, \theta+20, \theta^2+14) = \mathfrak{p}_{23}{}^2\mathfrak{q}_{23},$

$(47) = (47, 47\theta, 47\theta^2) = \mathfrak{p}_{47},$

$(53) = (53, \theta+16, \theta^2+9)(53, 53\theta, \theta^2+37\theta+43) = \mathfrak{p}_{53}\mathfrak{q}_{53},$

$(59) = (59, \theta+17, \theta^2+6)(59, \theta+46, \theta^2+8)(59, \theta+55, \theta^2+43)$
$= \mathfrak{p}_{59}\mathfrak{q}_{59}\mathfrak{r}_{59},$

the ideals on the right being all prime.

The sextic field $[\theta, \alpha]$ is normal and contains $[\theta]$, $[\theta']$, $[\theta'']$ as conjugate sub-fields. In fact

$$\left.\begin{aligned} 23\theta' &= 3\theta^2 - 16\theta - 2 + \alpha(-6\theta^2 + 9\theta + 4), \\ 23\theta'' &= -3\theta^2 - 7\theta + 2 + \alpha(6\theta^2 - 9\theta - 4). \end{aligned}\right\} \quad \ldots\ldots\ldots(31)$$

First to examine the sextic field as a relative-field of $[\sqrt{-23}]$, the discriminant of $b(\theta) = 0$, relative to $[\alpha]$, is

$$D_\alpha(\theta) \equiv -23 \text{ and } 23\,\mathfrak{o}_\alpha = \mathfrak{b}_{23}^{\,2}.$$

The expansion of Chap. v is

$$z^3 - z - 1 \equiv (z-10)^2(z-3) + 23(z^2 - 7z + 13)$$

$$\equiv \begin{bmatrix}(z-10)^2 & \\ & +16\,.\,23\,(z-10) + 16\,.\,23\end{bmatrix}\begin{bmatrix}z-3 & \\ & +8\,.\,23\end{bmatrix} \pmod{23^2}.$$

Noting that $23 = -(1-2\alpha)^2$, and

$$(z-10)^2 + 16\,.\,23 = (z-10)^2 - 16(1-2\alpha)^2$$

$$\equiv (z - 14 + 8\alpha)(z - 6 - 8\alpha)\ (\text{modd}\ \{z-10, \mathfrak{b}_{23}\}^3),$$

the third dissection separates the prime ideal factors of $\mathfrak{b}_{23}$; whence

$$\mathfrak{O} = \left[\mathfrak{o}_\alpha,\ \mathfrak{o}_\alpha\theta,\ \frac{\mathfrak{b}_{23}(\theta-10)(\theta-3)}{23}\right]$$

$$= [1, \alpha;\ (1, \alpha)\theta;\ \tfrac{1}{23}(23, \alpha+11)(\theta-10)(\theta-3)].$$

Also $\mathfrak{D}_\alpha = D_\alpha(\theta)\,\mathfrak{o}_\alpha \div \mathfrak{b}_{23}^{\,2} = (1)$, $D = D_\alpha{}^3 N(\mathfrak{D}_\alpha) = -23^3$, and

$$(23, \alpha + 11) = \mathfrak{P}_{23}\,\mathfrak{Q}_{23}\,\mathfrak{R}_{23} \text{ with}$$

$$\mathfrak{P}_{23} = [23, \alpha+11;\ (1, \alpha)(\theta-3);\ \tfrac{1}{23}(23, \alpha+11)(\theta-10)(\theta-3)],$$

$$\mathfrak{Q}_{23} = [23, \alpha+11;\ (1, \alpha)(\theta-10);\ \tfrac{1}{23}(23, \alpha+11)(\theta-14+8\alpha)(\theta-3)],$$

$$\mathfrak{R}_{23} = [23, \alpha+11;\ (1, \alpha)(\theta-10);\ \tfrac{1}{23}(23, \alpha+11)(\theta-6-8\alpha)(\theta-3)].$$

The factorisation of a prime ideal $(p, p\alpha)$ or $(p, \alpha+c)$ will be adequately illustrated by taking $p = 53$, 47 and 59.

(i) When $(-23|p) = -1$, $(p, p\alpha)$ is a prime ideal. Further, $b(z)$ has a linear and an irreducible quadratic divisor (mod p), but the quadratic divisor is reducible (mod $p\mathfrak{o}_\alpha$), e.g. $(-23|53) = -1$,

$$z^3 - z - 1 \equiv (z+16)(z^2 + 37z + 43) \pmod{53} \text{ in } [1] \text{ and}$$

$$\equiv (z+16)(z+10+17\alpha)(z+27+36\alpha) \pmod{\mathfrak{b}_{53}} \text{ in } [\alpha].$$

It follows that $(53) = \mathfrak{P}_{53}\,\mathfrak{Q}_{53}\,\mathfrak{R}_{53}$ where

$$\mathfrak{P}_{53} = [53(1, \alpha);\ (1, \alpha)(\theta+16);\ \tfrac{1}{23}(23, \alpha+11)(\theta^2 - 13\theta + 490)],$$

$$\mathfrak{Q}_{53} = [53(1, \alpha);\ (1, \alpha)(\theta+10+17\alpha);\ \tfrac{1}{23}(23, \alpha+11)(\theta^2 + 355\theta + 467)],$$

$$\mathfrak{R}_{53} = [53(1, \alpha);\ (1, \alpha)(\theta+27+36\alpha);\ \tfrac{1}{23}(23, \alpha+11)(\theta^2 + 355\theta + 467)].$$

(ii) Next when $(-23|p)=+1$, $(p)=\mathfrak{b}_p\mathfrak{c}_p$. If $b(z)$ is irreducible (mod p) in $[1]$ it is also irreducible (mod $\mathfrak{b}_p$) and (mod $\mathfrak{c}_p$) in $[\alpha]$, and $\mathfrak{b}_p\mathfrak{O}$, $\mathfrak{c}_p\mathfrak{O}$ are prime ideals of the third degree in the sextic field,

$$\text{e.g. } (47)=\mathfrak{P}_{47}\mathfrak{Q}_{47} \text{ with}$$

$$\mathfrak{P}_{47}=[47, \alpha+13; (47, \alpha+13)\,\theta; \tfrac{1}{23}(47, \alpha+13)(23, \alpha+11)(\theta^2-13\theta+30)],$$
$$\mathfrak{Q}_{47}=[47, \alpha+33; (47, \alpha+33)\,\theta; \tfrac{1}{23}(47, \alpha+33)(23, \alpha+11)(\theta^2-13\theta+30)].$$

(iii) In other cases where $(-23|p)=+1$, $b(z)$ is congruent to the product of three linear divisors (mod p) and each of the prime ideals $\mathfrak{b}_p$, $\mathfrak{c}_p$ is the product of three prime ideals in the relative field. Taking $p=59$,

$$(59, \alpha+26)=\mathfrak{P}_{59}\mathfrak{Q}_{59}\mathfrak{R}_{59}, \quad (59, \alpha+32)=\mathfrak{L}_{59}\mathfrak{M}_{59}\mathfrak{N}_{59},$$
$$\mathfrak{P}_{59}=[59, \alpha+26; (1, \alpha)(\theta+17); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta+375)],$$
$$\mathfrak{Q}_{59}=[59, \alpha+26; (1, \alpha)(\theta+46); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta-177)],$$
$$\mathfrak{R}_{59}=[59, \alpha+26; (1, \alpha)(\theta+55); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta-200)],$$
$$\mathfrak{L}_{59}=[59, \alpha+32; (1, \alpha)(\theta+17); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta+375)],$$
$$\mathfrak{M}_{59}=[59, \alpha+32; (1, \alpha)(\theta+46); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta-177)],$$
$$\mathfrak{N}_{59}=[59, \alpha+32; (1, \alpha)(\theta+55); \tfrac{1}{23}(23, \alpha+11)(\theta^2-13\theta-200)].$$

Oltramare (*l.c.*, 314) has given a criterion distinguishing those primes p for which an irreducible cubic is irreducible (mod p) from those for which it is congruent (mod p) to the product of three linear divisors.

Next considering $[\theta, \alpha]$ as a super-field of $[\theta]$,

$$\mathfrak{o}\,D_\theta(\alpha)=23\,\mathfrak{o}=\mathfrak{p}_{23}{}^2\,\mathfrak{q}_{23},$$

and the expansion of Chap. v is

$$\left.\begin{aligned} z^2-z+6 \equiv (z+11)^2 & \\ & -20(z+11)+6\,.\,23 \end{aligned}\right\} (\text{mod } 23^2).$$

Noting that $$(\theta-10)^2+16\,.\,23\equiv 0\ (\mathfrak{p}_{23}{}^3)$$
and $$(z+11)^2+6\,.\,23\equiv(z+11)^2-9\,(\theta-10)^2$$
$$\equiv(z-3\theta-5)(z+3\theta-19)\ (\text{modd } \{z-10,\ \mathfrak{p}_{23}\}^3),$$
the divisors of the third dissection are distinct. Hence

$$\mathfrak{O}=\left[\mathfrak{o}_\theta,\ \frac{\mathfrak{o}_\theta(\alpha+11)}{\mathfrak{p}_{23}}\right]=\left[\mathfrak{o}_\theta,\ \frac{\mathfrak{p}_{23}\,\mathfrak{q}_{23}(\alpha+11)}{23}\right]$$
$$=[1, \theta, \theta^2; \tfrac{1}{23}(23, 23\theta, \theta^2-13\theta+30)(\alpha+11)],$$
$$\mathfrak{D}_\theta=\mathfrak{o}\,D_\theta(\alpha)\div\mathfrak{p}_{23}{}^2=\mathfrak{q}_{23}$$

and $D=D_\theta{}^2\,N(\mathfrak{q}_{23})=-23^3$ as before.

From the same expansion $\mathfrak{p}_{23}=\mathfrak{Q}_{23}\mathfrak{R}_{23}$ with

$$\mathfrak{Q}_{23}=[23, \theta-10, \theta^2-8; \tfrac{1}{23}(23, 23\theta, \theta^2-13\theta+30)(\alpha+3\theta-19)],$$
$$\mathfrak{R}_{23}=[23, \theta-10, \theta^2-8; \tfrac{1}{23}(23, 23\theta, \theta^2-13\theta+30)(\alpha-3\theta-5)].$$

Since $z^2-z+6\equiv(z+11)^2\ (\text{mod } 23)$, $\mathfrak{q}_{23}=\mathfrak{P}_{23}{}^2$ where

$$\mathfrak{P}_{23}=[23, \theta-3, \theta^2-9; \tfrac{1}{23}(23, 23\theta, \theta^2-13\theta+30)(\alpha+11)].$$

(i)′ When $(-23 \mid p) = -1$, $(p) = \mathfrak{p}_p \mathfrak{q}_p$ in $[\theta]$ and $a(z)$ is irreducible (mod p), and therefore (mod $\mathfrak{p}_p$) and (mod $\mathfrak{q}_p$) in $[1]$. In $[\theta]$, $a(z)$ is irreducible (mod $\mathfrak{p}_p$) but reducible (mod $\mathfrak{q}_p$).

Taking $p = 53$, $\mathfrak{p}_{53} = (53,\ \theta + 16,\ \theta^2 + 9)$,

$$\mathfrak{P}_{53} = \mathfrak{p}_{53}\mathfrak{O} = [53,\ \theta + 16,\ \theta^2 + 9;$$
$$\tfrac{1}{23}(53,\ \theta + 16,\ \theta^2 + 9)(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 11)].$$

Since $\qquad a(z) \equiv (z + 25\theta + 38)(z + 28\theta + 14) \pmod{\mathfrak{q}_{53}}$

it follows without difficulty that $\mathfrak{O}\mathfrak{q}_{53} = \mathfrak{Q}_{53}\mathfrak{R}_{53}$ where

$$\mathfrak{Q}_{53} = [53,\ 53\theta,\ \theta^2 + 37\theta + 43;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 1840\theta + 356)],$$
$$\mathfrak{R}_{53} = [53,\ 53\theta,\ \theta^2 + 37\theta + 43;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 10350\theta + 862)].$$

(ii)′ Next when (p) is a prime ideal in $[\theta]$, $(-23 \mid p) = +1$ and $a(z)$ is reducible (mod p). Thus $(p) = (47) = \mathfrak{P}_{47}\mathfrak{Q}_{47}$,

$$\mathfrak{P}_{47} = \mathfrak{b}_{47}\mathfrak{O} = [47,\ 47\theta,\ 47\theta^2;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 1000)],$$
$$\mathfrak{Q}_{47} = \mathfrak{c}_{47}\mathfrak{O} = [47,\ 47\theta,\ 47\theta^2;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 80)].$$

(iii)′ Again when $(p) = \mathfrak{p}_p \mathfrak{q}_p \mathfrak{r}_p$ in $[\theta]$, $(-23 \mid p) = 1$ and $(p) = \mathfrak{b}_p \mathfrak{c}_p$ in $[\alpha]$. Taking $p = 59$, the six prime ideal factors are

$$\mathfrak{P}_{59} = (\mathfrak{p}_{59},\ \mathfrak{b}_{59}) = [59,\ \theta + 17,\ \theta^2 + 6;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 793)],$$
$$\mathfrak{L}_{59} = (\mathfrak{p}_{59},\ \mathfrak{c}_{59}) = [59,\ \theta + 17,\ \theta^2 + 6;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 563)],$$
$$\mathfrak{Q}_{59} = (\mathfrak{q}_{59},\ \mathfrak{b}_{59}) = [59,\ \theta + 46,\ \theta^2 + 8;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 793)],$$
$$\mathfrak{M}_{59} = (\mathfrak{q}_{59},\ \mathfrak{c}_{59}) = [59,\ \theta + 46,\ \theta^2 + 8;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 563)],$$
$$\mathfrak{R}_{59} = (\mathfrak{r}_{59},\ \mathfrak{b}_{59}) = [59,\ \theta + 55,\ \theta^2 + 43;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 793)],$$
$$\mathfrak{N}_{59} = (\mathfrak{r}_{59},\ \mathfrak{c}_{59}) = [59,\ \theta + 55,\ \theta^2 + 43;\ \tfrac{1}{23}(23,\ 23\theta,\ \theta^2 - 13\theta + 30)(\alpha + 563)],$$

and, evidently, $\mathfrak{p}_{59}\mathfrak{O} = \mathfrak{P}_{59}\mathfrak{L}_{59}$, $\mathfrak{b}_{59}\mathfrak{O} = \mathfrak{P}_{59}\mathfrak{Q}_{59}\mathfrak{R}_{59}$ etc.

It is readily seen that the ideals $\mathfrak{P}_p$ in (i) and (i)′ etc. are identical. Every prime ideal in the normal field, other than $\mathfrak{P}_{23}$, $\mathfrak{Q}_{23}$, $\mathfrak{R}_{23}$, is a prime ideal of the first degree in one of the fields $[\theta,\ \alpha]$, $[\theta]$, $[\theta']$, $[\theta'']$, $[\alpha]$.

Hilbert has given a discussion of certain sub-fields of a normal field and the groups with which they are associated. In the present example the *group of inertia* [*Trägheitsgruppe*] of each of the prime ideals $\mathfrak{P}_{23}$, $\mathfrak{Q}_{23}$, $\mathfrak{R}_{23}$ is identical with its *resolving group* [*Zerlegungsgruppe*]. The corresponding *resolving fields* are $[\theta]$, $[\theta']$, $[\theta'']$ and

$$\mathfrak{P}_{23}{}^2 = \mathfrak{q}_{23}\mathfrak{O},\quad \mathfrak{Q}_{23}{}^2 = \mathfrak{q}_{23}{}'\mathfrak{O},\quad \mathfrak{R}_{22}{}^2 = \mathfrak{q}_{23}{}''\mathfrak{O}.$$

CHAPTER XIV

EXTENSIONS OF PREVIOUS RESULTS

74. When the third dissection fails to separate the prime ideal factors of p the reasoning of § 49 still shews that $\varpi(\theta)^{\pi-\nu}\chi_k(\theta)$ is divisible by each prime ideal factor of $(p, \varpi(\theta))$ at least as often as p^k. There exist cases however in which an integral polynomial

$$C(\theta) \equiv \theta^c + c_1\theta^{c-1} + \dots + c_c,$$

of lower rank than $\varpi(\theta)^{\pi-\nu}\chi_k(\theta)$ is divisible by each of these prime ideals as highly as p^k, so it cannot, in general, be inferred that

$$\chi_k(\theta) = \varpi(\theta)^{\pi-\nu}\chi_k(\theta).$$

A simple numerical example is given by

$$\theta^9 - 2 \,.\, 3^3 = 0, \quad D(\theta) = 2^8 \,.\, 3^{42}.$$

The $(3, z)$ diagram has one edge joining (9, 0), (0, 3), and since

$$z^9 - 2 \,.\, 3^3 \equiv (z^3 + 3)^3 \ (\text{modd}\ \{\sqrt[3]{3}, z\}^{10})$$

the divisors of the third dissection are identical. Had they been distinct, the rank of $\chi_2(\theta)$ would have been six, as in Chap. IX, whereas the number

$$\omega = \theta^2(\theta^3 + 3)/9,$$

of the fifth rank, is integral since

$$\omega^9 - 6\omega^6 - 4 = 0.$$

It is evident that an unlimited quantity of numerical examples can be constructed with similar features. A numerical example could probably be constructed to defy any finite number of dissections of the type contained in Chaps. VI–VIII.

75. In certain cases of failure the difficulty can be removed by taking a new irreducible divisor

$$\kappa(z) \text{ which} \equiv \varpi(z) \ (\text{mod}\, p).$$

Thus in the field given by

$$a(\theta) = \theta^2 + 28 = 0,$$

with $\varpi(z) = z$ and $p = 2$ there is one edge, joining (2, 0) to (0, 2), and since

$$a(z) \equiv (z + 2)^2 \ (\text{modd}\ (2, z)^3)$$

the third dissection fails. But taking $w \equiv z + 2$ as a new irreducible divisor,

$$a(z) = w^2 - 2^2 . w + 2^5$$

and the second dissection now separates the prime ideal factors of (2).

76. Another example is given by

$$a(\theta) \equiv \theta^3 - 5 = 0$$

regarded as a relative-field of $[\sqrt[3]{7}]$. Taking

$$a^3 - 7 = 0,$$

$$\mathfrak{o} = (1, a, a^2), \quad \mathfrak{p}_3 = (3, a - 1, a^2 - 1), \quad (3) = \mathfrak{p}_3^3,$$

$$(5) = (5, a - 3, a^2 - 4)(5, 5a, a^2 + 3a + 4) = \mathfrak{p}_5 \mathfrak{q}_5,$$

also

$$\mathfrak{o} D(\theta) = \mathfrak{p}_3^9 \mathfrak{p}_5^2 \mathfrak{q}_5^2.$$

Since

$$a(z) = (z + 1)^3 - 3(z + 1)^2 + 3(z + 1) - 6,$$

the $\mathfrak{p}_3$-diagram contains one edge joining (3, 0) to (0, 3) and the third dissection fails. Evidently $\theta + 1 \equiv 0 \pmod{\mathfrak{p}_3}$, so putting

$$\lambda = \frac{2(\theta + 1)}{a - 1},$$

$$b(\lambda) \equiv \lambda^3 - (a^2 + a + 1)\lambda^2 + (a^2 + 3a + 5)\lambda - (6a^2 + 10a + 22) = 0,$$

and

$$b(z) \equiv (z + 1)^3 \pmod{\mathfrak{p}_3}.$$

But since

$$\begin{aligned} b(z) = (z + 1)^3 &- \{3(a - 1) + 2 . 3 + (a - 1)^2\}(z + 1)^2 \\ &+ \{2 . 3^2 + 3(a - 1)^2 + 11(a - 1)\}(z + 1) - 10 . 3(a - 1) - 17 . 3 - 8(a - 1)^2 \end{aligned}$$

the second dissection now separates the prime ideal factors of $\mathfrak{p}_3$ in the relative-field.

77. In certain cases a suitable transformation

$$\lambda = x_1\theta^{n-1} + x_2\theta^{n-2} + \ldots + x_n$$

gives a new representation of the field, $b(\lambda) = 0$, in which the third dissection, at furthest, separates the prime ideal factors of p. Taking

$$\theta^4 - 4(4n + 3) = 0$$

a fitting transformation for $p = 2$ is

$$\lambda = \tfrac{1}{4}(\theta^2 + 2\theta + 2):$$

cf. §§ 104–5, below.

The following questions present themselves in this connection:

(i) Can every number-field be defined by an equation $b(\lambda) = 0$ which makes

$$(1, \lambda, \lambda^2, \ldots \lambda^{n-1})$$

an integral basis

(ii) Failing (i), can it always be represented in such a way that

$$\mathfrak{o} \equiv (1, \lambda, \lambda^2, \ldots \lambda^{n-1}) \text{ (bas } p),$$

p being an *assigned* prime number?

(iii) Does every field possess a representation $b(\lambda) = 0$ such that no two distinct prime ideal factors of an assigned prime number p divide the same irreducible divisor of $b(\lambda) \pmod p$?

A suitably chosen numerical example is sufficient to shew that each of these questions must be answered in the negative. The example is given by

$$\theta^4 - 17 = 0$$

for which $$\mathfrak{o} = \left(1, \theta, \frac{\theta^2+1}{2}, \frac{\theta^3+\theta^2+\theta+1}{4}\right) = (1, \theta, \omega_2, \omega_3)$$

and $(2) = \mathfrak{p}_1{}^2 \mathfrak{p}_2 \mathfrak{p}_3$ with

$$\mathfrak{p}_1 = (2, 1+\theta, \omega_2, \omega_3), \quad \mathfrak{p}_2 = (2, 1+\theta, 1+\omega_2, \omega_3),$$
$$\mathfrak{p}_3 = (2, 1+\theta, 1+\omega_2, 1+\omega_3).$$

Taking a transformed equation $b(\lambda) = 0$, the only possible linear divisors of $b(z) \pmod 2$ are z and $z+1$. Were

$$\mathfrak{o} \equiv (1, \lambda, \lambda^2, \lambda^3) \text{ (bas 2)},$$

(2) could only have two distinct prime ideal factors of the first degree, their partial bases being

$$(2, \lambda, \lambda^2, \lambda^3) \text{ and } (2, \lambda+1, \lambda^2+1, \lambda^3+1) \text{ (bas 2)}.$$

Since (2) has three distinct prime ideal factors of the first degree, the results stated follow at once.

It can, however, be proved that there always exists a transformation separating the prime ideal factors of (p) at the second dissection. Taking

$$\left.\begin{aligned} &(p) = \mathfrak{p}_1{}^{f_1} \mathfrak{p}_2{}^{f_2} \ldots \mathfrak{p}_m{}^{f_m}, \\ &(\lambda) = \mathfrak{p}_1{}^{g_1} \mathfrak{p}_2{}^{g_2} \ldots \mathfrak{p}_m{}^{g_m} \mathfrak{a}, \quad (\mathfrak{a}, p\mathfrak{o}) = 1,\ dv(f_i, g_i) = 1, \end{aligned}\right\} \ldots (32)$$

with the ratios $f_1/g_1, \ldots f_m/g_m$ all distinct, there is one diagram, of m edges, for the prime p, whence the result stated.

78. In certain cases of failure of the third dissection another method is effective. For convenience it will be supposed that there is only one edge to the p-diagram. The field, of degree nu, is defined by

$$a(\theta) \equiv \theta^{nu} + a_1 p^v \theta^{(n-1)u} + \ldots + a_n p^{nv} - f(\theta) = 0$$

where $dv(u, v) = dv(p, a_n) = 1$ and $f(\theta)$ is a sum of terms represented by nodes on

$$xv + yu = nuv + w, \quad w > 0.$$

Evidently θ^u/p^v is a complex integer. A number $\psi(\theta)$ will be said to be isobaric with p^τ (mod p) when

$$\left.\begin{array}{l}\text{(i) } \psi(\theta)/p^\tau \text{ is integral and} \\ \text{(ii) } N[\psi(\theta)] \equiv 0 \pmod{p^{nu\tau}} \text{ exactly}\end{array}\right\} \qquad \ldots\ldots\ldots\ldots(33).$$

When τ is a rational fraction $\psi(\theta)/p^\tau$ does not belong to $[\theta]$. If $\psi_1(\theta)$ is isobaric with p^{τ_1}, $\psi_2(\theta)$ with p^{τ_2}, $\psi_3(\theta)$ with p^{τ_3} and $\tau_1 < \tau_2 \leqslant \tau_3$ the number

$$A_1\psi_1(\theta) + A_2\psi_2(\theta) + A_3\psi_3(\theta)$$

is evidently isobaric with p^{τ_1} (mod p).

Taking t, $t\tau$ to be coprime integers each prime ideal factor of p divides $\psi(\theta)^t$, $p^{t\tau}$ equally often. It may happen that, for a particular value of A, with

$$\chi(\theta) = \psi(\theta)^t + Ap^{t\tau},$$

$\chi(\theta)/p^{t\tau+\sigma}$ is integral and $N[\chi(\theta)] \equiv 0 \pmod{p^{nu(t\tau+\sigma)}}$ exactly, σ being > 0. More generally (when the degrees of the prime ideal factors of p exceed unity),

$$\chi_1(\theta) \equiv \psi(\theta)^{st} + A_1p^{t\tau}\psi(\theta)^{(s-1)t} + \ldots + A_sp^{st\tau}$$

may be isobaric with $p^{st\tau+\sigma}$ for $\sigma > 0$. In the application to numerical cases σ is the reciprocal of a factor of nu. The field then only contains the members $p^{\lambda_i}\theta^{\mu_i}$, of rank $< rst$, isobaric with $\chi_1(\theta)^r$ (mod p). It may happen that

$$\chi_2(\theta) \equiv \chi_1(\theta)^r + B_1p^{\lambda_1}\theta^{\mu_1} + \ldots + B_cp^{\lambda_c}\theta^{\mu_c}$$

is isobaric with $p^{rst\tau+r\sigma+\rho}$ (mod p), for $\rho > 0$. Continuing the process, which necessarily stops after a finite number of such steps, the complete stem of $\mathbf{o}$ (bas p) is finally put down by noting that, for suitable values of x, y, z,

$$\theta^x\chi_1(\theta)^y\chi_2(\theta)^z$$

is isobaric with a higher power of p than any integer of lower rank. Detailed applications of this method are given in Chap. XV (VI and VIII) below.

CHAPTER XV

THE INTEGRAL BASIS OF $[\sqrt[n]{a}]$

79. The methods developed in the preceding pages will now be applied to determine the integral basis of the field defined by

$$\theta^n - a = 0.$$

Since $$D(\theta) = \pm N(n\theta^{n-1}) = \pm n^n a^{n-1}$$

it will be enough to obtain the partial basis of $\mathfrak{o}$ (bas p) whenever p divides n or a; in some cases the partial bases of prime ideal factors of p will be obtained as well. The following twenty-three cases require discussion.

I. $n = p, \quad dv(a, p) = 1.$

II. $n = p^h, \quad dv(a, p) = 1, \quad h > 1.$

III. $n = lp^h, \quad dv(a, p) = dv(l, p) = 1.$

IV. $a = p^m b, \quad dv(n, p) = dv(b, p) = 1.$

V. $a = p^m b, \quad dv(m, p) = dv(b, p) = 1.$

VI. $n = p^{k'}, \quad a = p^q b, \quad q = p^h, \quad dv(b, p) = 1, \quad b^p - b \not\equiv 0 \pmod{p^2}$, excluding the cases in which

$$h = 1, \quad k' \geq 2, \quad b^{p-1} - 1 \equiv p \pmod{p^2}.$$

VII. $n = p^{k'}, \quad a = p^{mq} b, \quad q = p^h, \quad dv(b, p) = dv(m, p) = 1$, $b^p - b \not\equiv 0 \pmod{p^2}$, excluding the cases where

$$h = 1, \quad k' \geq 2, \quad b^{p-1} - 1 \equiv mp \pmod{p^2}.$$

VIII. p odd, $\quad n = p^{1+k}, \quad a = p^{mp} b, \quad dv(b, p) = dv(m, p) = 1$, $b^{p-1} \equiv 1 + mp \pmod{p^2}, \quad m \geq 1, \quad k \geq 1.$

IX. $p = 2, \quad n = 2^{1+k}, \quad a = 2^{2m} b, \quad k \geq 1, \quad m \equiv 1 \pmod 2, \quad b \equiv 3 \pmod 8.$

X. $p = 2, \quad n = 2^{1+k}, \quad a = 2^{2m} b, \quad k \geq 1, \quad m \equiv 1 \pmod 2, \quad b \equiv 7 \pmod 8.$

XI. $n = lp^k, \quad a = p^{mq} b, \quad q = p^h, \quad k > h, \quad b^{p-1} - 1 \not\equiv 0 \pmod{p^2}$, $dv(b, p) = dv(l, p) = dv(m, p) = 1.$

XII. $n = lp^k, \quad a = p^{mq} b, \quad q = p^h, \quad k \leq h$, $dv(b, p) = dv(l, p) = dv(m, p) = 1.$

XIII—XXIII. $n = lp^{k'}, \quad a = p^{mq} b, \quad q = p^h, \quad k' > h \geq 1$, $b^{p-1} - 1 \equiv 0 \pmod{p^2}, \quad dv(b, p) = dv(l, p) = dv(m, p) = 1$: further subdivision in § 108 below.

80. I. $n=p, \quad dv\,(a, p)=1.$

The integers of $[\sqrt{a}]$ being well known, we assume that $p>2$. Since

$$z^p - a = (z-a)^p + {}^pC_1 a\,(z-a)^{p-1} + \ldots + {}^pC_{p-1} a^{p-1}\,(z-a) + a^p - a,$$

and ${}^pC_1, \ldots {}^pC_{p-1}$ contain the factor p once only, it is evident that, when $a^p - a \not\equiv 0 \pmod{p^2}$,

$$\mathfrak{o} \equiv (1, \theta, \ldots \theta^{p-1}) \text{ (bas } p)$$

and $(p)=\mathfrak{p}^p$ with

$$\mathfrak{p} \equiv (p, \theta - a, \ldots \theta^{p-1} - a^{p-1}) \text{ (bas } p).$$

But when $a^p - a \equiv 0 \pmod{p^2}$ the diagram has two edges which meet at $(p-1, 1)$ whence

$$(p) = \mathfrak{p}_1^{p-1}\mathfrak{p}_2 \text{ and } \theta - a \equiv 0 \pmod{\mathfrak{p}_1\mathfrak{p}_2}.$$

Since $(\theta - a)(\theta^{p-1} + a\theta^{p-2} + \ldots + a^{p-1}) = a - a^p \equiv 0 \pmod{p^2}$

the number $\omega = \frac{1}{p}(\theta^{p-1} + a\theta^{p-2} + \ldots + a^{p-1})$

is an integer which $\equiv 0 \pmod{\mathfrak{p}_1}$. Hence

$$\mathfrak{o} \equiv (1, \theta, \theta^2, \ldots \theta^{p-2}, \omega) \text{ (bas } p),$$
$$\mathfrak{p}_1 \equiv (p, \theta - a, \ldots \theta^{p-2} - a^{p-2}, \omega) \text{ (bas } p),$$
$$\mathfrak{p}_2 \equiv (p, \theta - a, \ldots \theta^{p-2} - a^{p-2}, p^{-1}(\theta - a)^{p-1}) \text{ (bas } p).$$

$D \equiv 0 \pmod{p^p}$ or $\pmod{p^{p-2}}$ exactly according as $a - a^p \not\equiv 0$ or $\equiv 0 \pmod{p^2}$.

When $p=3$ these results agree with those obtained by Woronoj. The integers of $[\sqrt[p]{a}]$ have previously been found by Westlund, who also distinguished the prime ideal factors of (p). Recent writers* have connected Fermat's last theorem with the congruence

$$a^{p-1} - 1 \equiv 0 \pmod{p^2}:$$

its roots have been tabulated by Beeger as far as $p = 199$.

81—84. II. $n = p^h, \quad dv\,(a, p) = 1, \quad h > 1;$

take also

$$a - a^n \equiv 0 \pmod{p^j} \text{ exactly}, \quad j \geqslant 1, \quad \pi = p^{-1}, \quad \zeta = p^{-j}.$$

81. Here $z^n - a \equiv (z-a)^n \pmod{p},$

$$z^n - a = (z-a)^n + \sum_{1,\, n-1} {}^nC_i a^i (z-a)^{n-i} + a^n - a,$$

$${}^nC_{n-n\pi^f} \equiv 0 \pmod{p^f} \text{ exactly}, \quad 1 \leqslant f \leqslant h-1,$$

and ${}^nC_i \equiv 0 \pmod{p^{f+1}}$ for $n - n\pi^f < i < n$.

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* L. E. Dickson's *History*, II, 764, 773.

Hence when p is odd and $j \leqslant h+1$ the diagram of squares has j edges whose slants are

$$(p^h - p^{h-1})^{-1}, \quad (p^{h-1} - p^{h-2})^{-1}, \quad \dots \quad (p^{h-j+2} - p^{h-j+1})^{-1}, \quad p^{-h+j-1},$$

and each prime ideal factor of (p) is of the first degree. In fact

$$\left.\begin{array}{r} (p) = \mathfrak{p}_1^{n-n\pi} \mathfrak{p}_2^{n\pi - n\pi^2} \dots \mathfrak{p}_{j-1}^{(p-1)np\zeta} \mathfrak{p}_j^{np\zeta}, \\ \theta - a \equiv 0 \ (\text{mod}\ \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_{j-1} \mathfrak{p}_j) \text{ exactly} \end{array}\right\} \dots\dots\dots\dots(34).$$

When $p = 2$ and $1 < j \leqslant h+1$ the last two edges run into one whose marked nodes represent terms

$$\equiv 2^{j-2} [(z-a)^{4n\zeta} + 2(z-a)^{2n\zeta} + 2^2] \ (\text{modd}\ \{z - a, \sqrt[2n\zeta]{2}\}^{2jn\zeta+1})$$

and the second factor is irreducible. Hence

$$\left.\begin{array}{r} (2) = \mathfrak{p}_1^{n\pi} \mathfrak{p}_2^{n\pi\pi} \dots \mathfrak{p}_{j-2}^{4n\zeta} \mathfrak{p}_{j-1}^{2n\zeta}, \\ \theta - a \equiv 0 \ (\text{mod}\ \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_{j-2} \mathfrak{p}_{j-1}) \text{ exactly} \end{array}\right\} \dots\dots\dots(35),$$

$\mathfrak{p}_{j-1}$ being of the second degree and the others of the first degree.

Whenever $j = 1$ there is only one edge of slant p^{-h}. In all such cases $(p) = \mathfrak{p}^n$ with

$$\mathfrak{o} \equiv (1, \theta, \theta^2, \dots \theta^{n-1}) \ (\text{bas}\ p),$$
$$\mathfrak{p} \equiv (p, \theta - a, \theta^2 - a^2, \dots \theta^{n-1} - a^{n-1}) \ (\text{bas}\ p).$$

Finally when $j > h+1$ (p being 2 or odd) the diagram has $h+1$ edges of slants

$$(p^h - p^{h-1})^{-1}, \quad (p^{h-1} - p^{h-2})^{-1}, \quad \dots \quad (p-1)^{-1}, \quad j - h,$$

and

$$\left.\begin{array}{r} (p) = \mathfrak{p}_1^{n-n\pi} \mathfrak{p}_2^{n\pi - n\pi\pi} \dots \mathfrak{p}_h^{p-1} \mathfrak{p}_{h+1}, \\ (\theta - a) \equiv 0 \ (\text{mod}\ \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_h \mathfrak{p}_{h+1}^{j-h}) \text{ exactly} \end{array}\right\} \dots\dots\dots(36),$$

all the prime ideals being of the first degree.

82. Putting $\eta_r(\theta) = \theta^{\rho(p-1)} + a^\rho \theta^{\rho(p-2)} + \dots + a^{\rho(p-1)}$

with
$$\rho = p^{h-r}, \quad r \leqslant h, \quad r < j,$$

and
$$\omega = \frac{\eta_1(\theta)\, \eta_2(\theta) \dots \eta_r(\theta)}{p^r} = \frac{a - \alpha^n}{p^r (\theta^\rho - \alpha^\rho)},$$

then
$$\theta^\rho - \alpha^\rho = \frac{a - \alpha^n}{p^r \omega}, \quad a = \left(\alpha^\rho + \frac{a - \alpha^n}{p^r \omega}\right)^{p^r},$$

i.e.
$$\omega^{p^r} - \sum_{1, p^r} {}^{p^r}C_i\, \alpha^{n - i\rho} \omega^{p^r - i} \frac{(a - \alpha^n)^{i-1}}{p^{ri}} = 0$$

which is the irreducible equation satisfied by ω. Each binomial coefficient being integral, ω is a complex integer provided

$$a - \alpha^n \equiv 0 \ (\text{mod}\ p^{r+1})$$

which is satisfied when $r<j$. Since ω is of rank p^h-p^{h-r}, the same as $\phi_r(\theta)$, it is sufficient to take the stem of $\mathfrak{o}$ (bas p) to be

$$1,\ \frac{\eta_1(\theta)}{p},\ \frac{\eta_1(\theta)\,\eta_2(\theta)}{p^2},\ \dots\ \frac{\eta_1(\theta)\,\eta_2(\theta)\dots\eta_k(\theta)}{p^k}\ \dots(37),$$

$$\equiv 1,\ \Omega_1,\ \Omega_2,\ \dots\ \Omega_k,$$

where $k=j-1$ in (34) and (35), but $k=h$ in (36).

The discriminant D is divisible by p to the power

$$hp^h-2\sum_{1,\,k-1} i\,(p^{h-i}-p^{h-i-1})-2kp^{h-k}$$

which

$$=hp^h-2p^{h-k}\,(p^k-1)/(p-1).$$

83. It being readily seen that

$$\eta_r(z)\equiv(z-a)^{\rho(p-1)}\ (\mathrm{mod}\ p)\ \text{ and }\ z^\rho-a^\rho\equiv(z-a)^\rho\ (\mathrm{mod}\ p),$$

each of the prime ideals $\mathfrak{p}_1,\ \mathfrak{p}_2,\ \dots\ \mathfrak{p}_{r-1}$ divides $\eta_r(\theta)$, $\theta^\rho-a^\rho$ just as often as $(\theta-a)^{\rho(p-1)}$, $(\theta-a)^\rho$ respectively. Hence, when $s<k$, $\mathfrak{p}_s$ divides

$$\eta_{s+1}(\theta),\ \dots\ \eta_k(\theta),\ \theta^{p^{h-k}}-a^{p^{h-k}}$$

just

$$p^{h-s-1}(p-1),\ \dots\ p^{h-k}(p-1),\ p^{h-k}$$

times respectively. Since also

$$\eta_{s-i}(z)\equiv\eta_s(z)^{p^i}+pa^{p^{h-s+i}(p-1)}\ (\mathrm{modd}\ \{p^2,\,p\,(z-a)\}),$$

$\mathfrak{p}_s$ divides $\eta_{s-i}(\theta)$ just as often as p, i.e. $p^{h-s}(p-1)$ times. From the identity

$$\eta_1(\theta)\,\eta_2(\theta)\dots\eta_k(\theta)\,(\theta^{p^{h-k}}-a^{p^{h-k}})=a-a^n\ \dots\dots\dots(38),$$

it now follows that $\mathfrak{p}_s$ divides $\eta_s(\theta)$ just

$$[(j-s+1)\,(p-1)-1]\,p^{h-s}\ \text{times.}$$

Accordingly

$$\eta_1(\theta)\,\eta_2(\theta)\dots\eta_s(\theta)$$

contains the factor $\mathfrak{p}_s$ to the power

$$[(j-s+1)\,(p-1)-1]\,p^{h-s}+(s-1)\,(p-1)\,p^{h-s}$$

which

$$=[j\,(p-1)-1]\,p^{h-s}\ \text{and}\ \geqslant k\,(p-1)\,p^{h-s}.$$

When p is odd the prime ideal $\mathfrak{p}_k$ divides each of the numbers $\eta_1(\theta)$, $\eta_2(\theta)$, $\dots$ $\eta_{k-1}(\theta)$ just as often as p, and $\theta^{p^{h-k}}-a^{p^{h-k}}$ just p^{h-k} times. Hence, after (38), $\mathfrak{p}_k$ divides $\eta_k(\theta)$ just

$$[(j-k+1)\,(p-1)-1]\,p^{h-k}$$

which

$$>(p-1)\,p^{h-k}\ \text{times.}$$

Hence

$$\eta_1(\theta)\,\eta_2(\theta)\dots\eta_k(\theta)\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_k^{1+k(p-1)\,p^{h-k}}).$$

Again, p being 2 or odd and $r < s \leqslant k \leqslant h$,

$$\eta_r(z) \equiv \eta_{r+1}(z)^p + p \ (\text{modd}\ \{p^2, p(z-a)\}),$$

whence $\eta_{s-1}(\theta) \equiv \eta_{s-2}(\theta) \equiv \ldots \equiv \eta_1(\theta) \equiv p \ (\text{mod}\ \mathfrak{p}_s^{1+(p-1)p^{h-s}}$

and $$\eta_1(\theta)\,\eta_2(\theta) \ldots \eta_r(\theta) \equiv p^r \ (\text{mod}\ \mathfrak{p}_s^{1+r(p-1)p^{h-s}}).$$

In the case of $\mathfrak{p}_{k+1} (= \mathfrak{p}_{h+1})$ in (36)

$$\eta_p(z) \equiv z^{p-1} + az^{p-2} + \ldots + a^{p-1} \equiv (z-a)^{p-1} \ (\text{modd}\ \{p^2, p(z-a)\})$$

so, when p is odd, $\eta_h(\theta) \equiv 0 \ (\text{mod}\ \mathfrak{p}_{h+1}^2)$,

and, since $\eta_{r-1}(z) \equiv \eta_r(z)^p \ (\text{modd}\ \{p^2, p(z-a)\})$,

$$\eta_{h-1}(\theta) \equiv \eta_{h-2}(\theta) \equiv \ldots \equiv \eta_1(\theta) \equiv 0 \ (\text{mod}\ \mathfrak{p}_{h+1}^2),$$

$$\eta_1(\theta)\,\eta_2(\theta) \ldots \eta_r(\theta) \equiv 0 \ (\text{mod}\ \mathfrak{p}_{h+1}^{r+1}).$$

But, when $p = 2$,

$$\eta_h(\theta) = \theta + a = (\theta - a) + 2a \equiv 2 \ (\text{mod}\ \mathfrak{p}_{h+1}^2),$$

$$\eta_{h-1}(\theta) = \theta^2 + a^2 = (\theta - a)(\theta + a) + 2a^2 \equiv 2 \ (\text{mod}\ \mathfrak{p}_{h+1}^2),$$

hence $$\eta_r(\theta) \equiv 2 \ (\text{mod}\ \mathfrak{p}_{h+1}^2)$$

and $$\eta_1(\theta)\,\eta_2(\theta) \ldots \eta_r(\theta) \equiv 2^r \ (\text{mod}\ \mathfrak{p}_{h+1}^{r+1}).$$

Similarly in the case of $\mathfrak{p}_{k+1} \equiv \mathfrak{p}_j$ of (34)

$$\eta_{j-1}(\theta) \equiv \eta_{j-2}(\theta) \equiv \ldots \equiv \eta_1(\theta) \equiv 0 \ (\text{mod}\ \mathfrak{p}_j^2),$$

$$\eta_1(\theta)\,\eta_2(\theta) \ldots \eta_r(\theta) \equiv 0 \ (\text{mod}\ \mathfrak{p}_j^{r+1}), \quad r \leqslant j-1.$$

Thus, retaining only the stems of the partial bases,

$$\left.\begin{array}{l}
\mathfrak{p}_1 \equiv (p, \theta - a, \Omega_1, \Omega_2, \ldots \Omega_k)\ (\text{bas}\ p) \\
\mathfrak{p}_s \equiv (p, \theta - a, \Omega_1 - 1, \ldots \Omega_{s-1} - 1, \Omega_s, \ldots \Omega_k)\ (\text{bas}\ p), \quad 1 < s < k \\
\mathfrak{p}_{j-1} \equiv (2, \theta - 1, \Omega_1 - 1, \ldots \Omega_{j-2} - 1, 2\Omega_{j-1}, (\theta - 1)\Omega_{j-1})(\text{bas}\ 2)\ \text{in}\ (35) \\
\mathfrak{p}_k \equiv (p, \theta - a, \Omega_1 - 1, \ldots \Omega_{k-1} - 1, \Omega_k)\ (\text{bas}\ p)\ \text{in}\ (34)\ \text{and}\ (36) \\
\mathfrak{p}_j \equiv (p, \theta - a, \Omega_1 - 1, \ldots \Omega_{j-1} - 1)\ (\text{bas}\ p)\ \text{in}\ (34) \\
\mathfrak{p}_{h+1} \equiv (p, \theta - a, \Omega_1 - 1, \ldots \Omega_h - 1)\ (\text{bas}\ p)\ \text{in}\ (36),\ p\ \text{odd}, \\
\mathfrak{p}_{h+1} \equiv (2, \theta - a, \Omega_1 - 1, \ldots \Omega_h - 1)\ (\text{bas}\ 2)\ \text{in}\ (36)
\end{array}\right\} (39).$$

84. Noting that

$$a^n - a = a(a^{p-1} - 1)(a^{n-p} + a^{n-2p+1} + \ldots + 1)$$

and $$a^{n-p} + a^{n-2p+1} + \ldots + 1 \equiv p^{h-1} + p^{h-2} + \ldots + 1 \ (\text{mod}\ p)$$

(by Fermat's theorem), the congruence

$$a^n - a \equiv 0 \ (\text{mod}\ p^j)\ \text{exactly}$$

involves $$a^{p-1} - 1 \equiv 0 \ (\text{mod}\ p^j)\ \text{exactly}.$$

This last congruence has $p-1$ solutions which are incongruent (mod p). A solution of $a^{p-1}-1\equiv 0 \pmod{p^j}$ also satisfies $a^{p-1}-1\equiv 0 \pmod{p^{j-1}}$ and, if b is any solution of the latter, a root of the former is obtained by solving the linear congruence

$$0\equiv (b+xp^{j-1})^p-(b+xp^{j-1})\equiv b^p-b-xp^{j-1} \pmod{p^j}.$$

Solutions of $a^{p-1}-1\equiv 0 \pmod{p^j}$ are conveniently tabulated in Hensel's notation in which an entry

$$aa'a''\ \ldots,\quad 0<a<p,\quad 0\leqslant a',\ a'',\ \ldots<p,$$

means

$$a^{p-1}-1\equiv 0 \pmod{p}$$
$$(a+pa')^{p-1}-1\equiv 0 \pmod{p^2}$$
$$(a+pa'+p^2a'')^{p-1}-1\equiv 0 \pmod{p^3}$$

and so on. The solutions for $p=2$, 3, 5, 7, 11, 13, of which the last two sets have been calculated by Mr M. Cashmore of London, are as follows:

$$p=2,\quad 10000\quad 00000\quad 00\ldots;$$

$$p=3,\quad 22222\quad 22222\quad 22\ldots,\quad 10000\quad 00000\quad 00\ldots;$$

$$p=5,\quad \left\{\begin{array}{llllll} 21213 & 42303 & 22\ldots, & 33231 & 02141 & 22\ldots, \\ 44444 & 44444 & 44\ldots, & 10000 & 00000 & 00\ldots; \end{array}\right.$$

$$p=7,\quad \left\{\begin{array}{llllll} 34630 & 26243 & 44\ldots, & 42036 & 40423 & 22\ldots, \\ 24630 & 26243 & 44\ldots, & 52036 & 40423 & 22\ldots, \\ 66666 & 66666 & 66\ldots, & 10000 & 00000 & 00\ldots; \end{array}\right.$$

$$p=11,\quad \left\{\begin{array}{llllll} 2t491 & 23978 & tt\ldots, & 90619 & 87132 & 00\ldots, \\ 47952 & 98078 & 66\ldots, & 73158 & 12t32 & 44\ldots, \\ 8t987 & 4023t & 47\ldots, & 30123 & 6t870 & 63\ldots, \\ 52517 & 85t3t & 80\ldots, & 68593 & 25070 & 2t\ldots, \\ ttttt & ttttt & tt\ldots, & 10000 & 00000 & 00\ldots; \end{array}\right.$$

$$p=13,\quad \left\{\begin{array}{llllll} 26224 & 25873 & 10\ldots, & e6tt8 & t7459 & eT\ldots, \\ 4e697 & 24488 & 15\ldots, & 91635 & t8844 & e7\ldots, \\ 87eT7 & 7eTe4 & 46\ldots, & 55105 & 51018 & 86\ldots, \\ 3e697 & 24488 & 15\ldots, & t1635 & t8844 & e7\ldots, \\ 619t3 & 56441 & 36\ldots, & 10000 & 00000 & 00\ldots, \\ TTTTT & TTTT & TT\ldots, & 7e329 & 7688e & 96\ldots. \end{array}\right.$$

Col. Cunningham has recorded a few other solutions of $a^{p-1}-1\equiv 0 \pmod{p^j}$ with $j>2$ and $p\leqslant 43$.

As a numerical example take $\theta^{625}=a$. Putting

$$\begin{aligned}5\Omega_1 &= \theta^{500}+a^{125}\theta^{375}+\ldots+a^{500},\\ 25\Omega_2 &= \theta^{600}+a^{25}\,\theta^{575}+\ldots+a^{600},\\ 125\Omega_3 &= \theta^{620}+a^{5}\,\theta^{615}+\ldots+a^{620},\\ 625\Omega_4 &= \theta^{624}+a\,\theta^{623}+\ldots+a^{624},\end{aligned}$$

the stem of $\mathfrak{o}$ (bas 5) is as tabulated below:

$\mathfrak{o}\equiv$	when $a\equiv$	but $a\not\equiv$
1	1, 2, 3, 4 (mod 5)	1, 7, 18, 24 (mod 5^2)
1, Ω_1	1, 7, 18, 24 (mod 5^2)	1, 57, 68, 124 (mod 5^3)
1, Ω_1, Ω_2	1, 57, 68, 124 (mod 5^3)	1, 182, 443, 624 (mod 5^4)
1, Ω_1, Ω_2, Ω_3	1, 182, 443, 624 (mod 5^4)	1, 2057, 1068, 3124 (mod 5^5)
1, Ω_1, Ω_2, Ω_3, Ω_4	1, 2057, 1068, 3124 (mod 5^5)	

85. III. $n=lp^h,\quad dv(a,p)=dv(l,p)=1.$

The field can be defined by the equations

$$\theta^l=\alpha,\quad \alpha^q=a,\quad q=p^h;$$

it is a relative-field of that discussed in II. If $\mathfrak{o}$ is the integral basis of $[\alpha]$ the relative discriminant is

$$\mathfrak{o}D_\alpha(\theta)=\mathfrak{o}l^l\alpha^{l-1}$$

which has no prime ideal factor in common with (p).

Hence $\mathfrak{O}\equiv(1,\ \theta,\ \ldots\ \theta^{l-1})\,\mathfrak{o}$ (bas p)

and, arranged in ascending order of rank, its stem is

$$1,\ \frac{\eta_1(\theta^l)}{p},\ \ldots\ \frac{\eta_1(\theta^l)\,\eta_2(\theta^l)\,\ldots\,\eta_k(\theta^l)}{p^k}\ (\text{bas}\ p),$$

where k is defined as in II.

86. IV. $a=p^mb,\quad dv(n,p)=dv(b,p)=1;$

and V. $a=p^mb,\quad dv(m,p)=dv(b,p)=1.$

The case when m and n are both prime to p can be regarded as belonging to either of these which are solved in the same way.

Put $n=ut$, $m=vt$ with $dv(u,v)=1$; necessarily $dv(t,p)=1$. The field is defined by

$$\theta^u=\alpha,\quad \alpha^t=p^{vt}b,$$

and can be regarded as a super-field of $[\alpha]$. Since α/p^v is integral, and

$$D(\alpha/p^v)=t^tb^{t-1}\not\equiv 0\ (\text{mod}\ p),$$

$$\mathfrak{o}\equiv\left(1,\ \frac{\alpha}{p^v},\ \frac{\alpha^2}{p^{2v}},\ \ldots\ \frac{\alpha^{t-1}}{p^{(t-1)v}}\right)(\text{bas}\ p).$$

Passing to the relative-field, put

$$rm/n = e(r) + f(r)$$

with $e(r) \equiv I(rm/n)$, a rational integer, and $0 \leqslant f(r) < 1$. The numbers

$$1, \frac{\theta}{p^{e(1)}}, \frac{\theta^2}{p^{e(2)}}, \dots \frac{\theta^{u-1}}{p^{e(u-1)}}$$

are complex integers and, after § 78, it is readily seen that, if $\mathfrak{p}$ is a prime ideal factor of (p),

$$\mathfrak{O} \equiv \left(1, \frac{\theta}{p^{e(1)}}, \dots \frac{\theta^{u-1}}{p^{e(u-1)}}\right) \mathfrak{o} \text{ (bas } \mathfrak{p}).$$

If $n \equiv 0 \pmod{p^h}$ exactly, D is divisible by p just

$$nh + m(n-1) - \sum_{0,\, n-1} I\left(\frac{rm}{n}\right)$$

$$\equiv nh + m(n-1) - tuv(t-1) - 2t \sum_{0,\, v-1} I\left(\frac{rv}{u}\right)$$

times. Noting that, after an argument due to Eisenstein,

$$\sum_{0,\, v-1} I(rv/u) = \tfrac{1}{2}(u-1)(v-1),$$

$$D \equiv 0 \pmod{p^{t(u-1)}} \text{ exactly in IV, when } h = 0,$$

and $\qquad \equiv 0 \pmod{p^{nh+t(u-1)}}$ exactly in V.

87—94. VI.

$$n = p^{k'}, \quad a = p^q b, \quad q = p^h, \quad dv(b, p) = 1, \quad b^p \not\equiv b \pmod{p^2},$$

omitting an exceptional case to be noted later.

87. First to dispose of two extraneous cases.

When $k' \leqslant h$ the equation $\theta^n - a = 0$ becomes

$$(\theta/p^v)^n - b = 0, \quad v = p^{h-k'}$$

and, by II above,

$$\mathfrak{o} \equiv \left(1, \frac{\theta}{p^v}, \frac{\theta^2}{p^{2v}}, \dots \frac{\theta^{n-1}}{p^{(n-1)v}}\right) \text{ (bas } p).$$

Again when $h = 0$, $q = 1$, after IV above,

$$\mathfrak{o} \equiv (1, \theta, \theta^2, \dots \theta^{n-1}) \text{ (bas } p).$$

In all other cases, on writing

$$k' = h + k, \quad h \geqslant 1,$$

we have $\qquad \left(\frac{\theta^{p^k}}{p}\right)^{p^h} = b:$

it is convenient to put $\theta^{p^k} = p\zeta$ so that $\zeta^q = b$, also

$$\kappa = q^{-1}, \quad p^k = e, \quad \epsilon = e^{-1}, \quad e = q^f r', \quad 0 \leqslant f,$$

$$f \text{ integral}, \quad 1 \leqslant r' < q, \quad q = rr'.$$

88. Since θ^e/p, pb/θ^e are both integral, the numbers θ^e, p are isobaric (mod p) and each prime ideal factor of p divides p and θ just ew and w times respectively. Following out the process of § 78 we determine the greatest index ρ, and appropriate x_1, making

$$\frac{\theta^{e-\rho}(\theta^e - px_1)}{p^2} \text{ or } \left(\frac{bp}{\theta^e}\right)^{\frac{e-\rho}{e}} \cdot \frac{\theta^{e-\rho}(\theta^e - px_1)}{p^2} \text{ or } \frac{\theta^e - px_1}{p^{1+\rho\epsilon}}$$

a complex integer. Putting

$$\omega_1 = \frac{\theta^e - px_1}{p^{1+\rho\epsilon}} = \frac{\zeta - x_1}{p^{\rho\epsilon}},$$

we have

$$\omega_1 + p^{-\rho\epsilon}x_1 = p^{-\rho\epsilon}\zeta,$$

and

$$\omega_1^q + \sum_{1,\,q-1} {}^qC_i p^{-i\rho\epsilon} x_1^i \omega_1^{q-i} + p^{-q\rho\epsilon}(x_1^q - b) = 0.$$

The last coefficient is integral for

$$\rho = (q\epsilon)^{-1} = e\kappa \text{ when } x_1 = b$$

but never for a greater value of ρ since the congruence

$$x_1^q - b \equiv 0 \pmod{p^2}$$

is insoluble when $b^p - b \not\equiv 0 \pmod{p^2}$. Since also

$${}^qC_i \equiv 0 \pmod{p}, \quad 1 \leqslant i \leqslant q-1,$$

each of the numbers ${}^qC_i p^{-i\epsilon \cdot e\kappa}$, $1 \leqslant i \leqslant q-1$, is an integer: it follows that

$$\frac{\zeta - b}{p^\kappa} \equiv \frac{\theta^e - pb}{p^{1+\kappa}}$$

is integral and that $\theta^e - pb$ is isobaric with $p^{1+\kappa}$ (mod p).

When $f = 0$, $\theta^e - pb$ is isobaric with θ^y for no integral value of y. The integers

$$1, \frac{\theta^e}{p}, \frac{\theta^e(\theta^e - pb)}{p^2}$$

then form the first three terms of the stem of $\mathfrak{o}$ (bas p).

89. More generally, when $f \geqslant 1$, the numbers

$$\zeta - b, \quad \theta^{e\kappa}$$

are isobaric (mod p), so continuing the process of § 78 we determine the greatest exponent σ which makes

$$\omega_2 = \frac{\theta^e - pb + px_2\theta^{e\kappa}}{p^{1+\kappa+\sigma}} = \frac{\zeta - b + x_2\theta^{e\kappa}}{p^{\kappa+\sigma}}$$

a complex integer. On expansion,

$$\omega_2^q = p^{-1-q\sigma}\left[(\zeta - b)^q + x_2^q\theta^e + \sum_{1,\,q-1} {}^qC_i(\zeta - b)^i x_2^{q-i}\theta^{(q-i)e\kappa}\right],$$

and, each term under Σ being certainly divisible by p^2, it follows that

$$\omega_2{}^q = p^{-1-q\sigma}\left[(\zeta-b)^q + px_2\zeta\right] + \text{an integer} \quad \text{.........(40)}$$

so long as $1+q\sigma<2$ or $\sigma<\kappa$.

Noting that $b=[b+(\zeta-b)]^q$,

$$\begin{aligned}(\zeta-b)^q+px_2\zeta & \\ &= b-b^q-q(\zeta-b)\,b^{q-1}-\sum_{1,\,q-2}{}^qC_i(\zeta-b)^{q-i}b^i \\ &\quad + px_2(\zeta-b)+px_2b \\ &= p(\zeta-b)\left\{\frac{b^{q-1}-1}{p}-\frac{q}{p}\,b^{q-1}\right\}-\sum_{1,\,q-2}{}^qC_i(\zeta-b)^{q-i}b^i\end{aligned}$$

provided $x_2=(b^{q-1}-1)/p$. Since $\zeta-b$ is isobaric with p^κ (mod p), it follows that

$$(\zeta-b)^q+(b^{q-1}-1)\,\zeta$$

is isobaric with $p^{1+\kappa}$ except when

$$\frac{b^{q-1}-1}{p}-\frac{q}{p}\,b^{q-1}\equiv 0 \pmod{p} \quad \text{..................(41),}$$

in which event it is divisible by a higher power of p.

The congruence (41) gives

$$(1-q)\,b^{q-1}\equiv 1 \pmod{p^2}$$

and, since $b^{q-1}\equiv b^{p-1} \pmod{p^2}$, this involves

$$\left.\begin{array}{ll} & b^{p-1}\equiv 1+p \pmod{p^2} \\ \text{and} & p=q \\ \text{also} & f>0 \text{ already noted}\end{array}\right\} \text{.....................(42).}$$

An exceptional case mentioned in the enunciation of VI is defined by the conditions (42). Since, when $p=2$, $b\equiv 3$ (mod 4), all the fields for which $q=2$ are exceptional. We return to these exceptions in VIII when p is odd, in IX and X when $p=2$, and for the remaining part of VI assume that the conditions (42) are not satisfied.

In the present case $1+q\sigma=1+\kappa$

or $\sigma=\kappa^2$ which $<\kappa$, and it is sufficient to take

$$x_2=b_1=\text{least positive residue of } (b^{p-1}-1)/p \pmod{p} \text{...(43),}$$

an integer different from 0 and 1. The number

$$p^{1+\kappa+\kappa\kappa}\omega_2\equiv\theta^e-pb+pb_1\theta^{e\kappa} \quad \text{..................(44)}$$

is then isobaric with $p^{1+\kappa+\kappa\kappa}$ (mod p). When $f=1$ this number is isobaric with no integer of $[\theta]$ of rank $<e$ and

$$1,\ \frac{\theta^e}{p},\ \frac{\theta^{e-e\kappa}(\theta^e-pb+pb_1\theta^{e\kappa})}{p^2}$$

are the first three terms in the stem of $\mathfrak{o}$ (bas p).

90. When $f \geqslant 2$ the integers

$$\theta^e - pb + pb_1\theta^{e\kappa}, \quad p\theta^{e\kappa(1+\kappa)}$$

are isobaric (mod p): so we put

$$\omega_3 = \frac{\theta^e - pb + pb_1\theta^{e\kappa} + px_3\theta^{e\kappa(1+\kappa)}}{p^{1+\kappa+\kappa\kappa+\tau}} = \frac{\zeta - b + b_1\theta^{e\kappa} + x_3\theta^{e\kappa(1+\kappa)}}{p^{\kappa+\kappa\kappa+\tau}}$$

whence, on expansion,

$$\omega_3{}^q = \frac{(\zeta - b)^q + pb_1\zeta + x_3\theta^{e(1+\kappa)}}{p^{1+\kappa+q'\tau}} + \text{an integer} \quad \text{.........(45)},$$

provided $$1 + \kappa + q\tau < 2$$
or $$\tau < \kappa - \kappa^2.$$
Now $$(\zeta - b)^q + pb_1\zeta + x_3\theta^{e(1+\kappa)}$$

$$= p\left[(b_1 - q/p)(\zeta - b) + bx_3\theta^{e\kappa}\right] + M(p^{1+2\kappa}).$$

So taking $$b_2 \equiv b_1 - q/p \pmod{p},$$
i.e. $$\left.\begin{array}{l} b_2 = b_1 \text{ when } q > p \\ b_2 = b_1 - 1 \text{ when } q = p \end{array}\right\} \text{.....................(46)},$$
and $$x_3 b \equiv b_1 b_2 \pmod{p},$$
or $$x_3 = b^{p-2}b_1b_2,$$

a comparison with (44) shews that the numerator of the fraction in (45) is isobaric with $p^{1+\kappa+\kappa\kappa}$, that $\tau = \kappa^3$, and that

$$\theta^e - pb + pb_1\theta^{e\kappa} + pb^{p-2}b_1b_2\theta^{e\kappa(1+\kappa)}$$

is isobaric with $p^{(\kappa.3)}$ where

$$(\kappa . z) = 1 + \kappa + \kappa^2 + \ldots + \kappa^z.$$

91. Next assuming that

$$\theta^e - pb + pb_1\theta^{e\kappa} + \sum_{2,\, l} pb^{p-i}b_1b_2{}^{i-1}\theta^{e[(\kappa.i)-1]} \quad \text{.........(47)}$$

is isobaric with $p^{(\kappa.l+1)}$ (mod p) when $2 \leqslant l < f$, it readily follows by induction that the same result is true when $2 < l \leqslant f$. We cannot pass rationally to the next stage since $[\theta]$ does not contain the number

$$\theta^{e(\kappa.f+1)-e} \equiv \theta^{e(\kappa.f)-e} . \sqrt[q]{\theta^{r'}}.$$

92. Putting $$a = (\kappa . f + 1),$$
also $$\chi(\theta) = \theta^e - pb \text{ when } f = 0,$$

$$\chi(\theta) = \theta^e - pb + pb_1\theta^{e\kappa} \text{ when } f = 1,$$

$$\chi(\theta) = \theta^e - pb + pb_1\theta^{e\kappa} + \sum_{2,\, f} pb^{p-i}b_1b_2{}^{i-1}\theta^{e[(\kappa.i)-1]} \text{ when } f > 1,$$

the integer $\theta^s \chi(\theta)^t$ is isobaric with $p^{s\epsilon + t\alpha}$ (mod p) when s, t are positive. Further if

$$c = e\,[(\kappa\,.\,f) - 1], \text{ whence } e - (r-1)\,c > 0,$$

each of the numbers

$$1,\ \frac{\theta^e}{p},\ \frac{\theta^{e-c}\chi(\theta)}{p^2},\ \frac{\theta^{e-2c}\chi(\theta)^2}{p^3},\ \ldots\ \frac{\theta^{e-(r-1)c}\chi(\theta)^{r-1}}{p^r} \qquad \ldots\ldots(48)$$

is integral, whereas $\theta^{e-ic-1}\chi(\theta)^i/p^{i+1}$ is fractional when $1 \leqslant i \leqslant r-1$. No two of the last r integers in (48) being isobaric (mod p), these $r+1$ numbers are the first $r+1$ terms in the stem of $\mathfrak{o}$ (bas p). They form the complete stem when

$$r' = 1, \text{ i.e. when } e = q^r,\ n = q^{f+1},$$

the rank of the last one then being $e\,(q-1)+1$.

93. More generally when $1 < r' < q$ the number

$$\chi'(\theta) \equiv \theta^e - pb + pb_1\theta^{e\kappa} + p \sum_{2,\,f+1} b^{p-i} b_1 b_2^{\,i-1} \theta^{e\,[(\kappa\,.\,i)-1]},$$

which lies in $[\sqrt[q]{\theta^r}]$, is isobaric with $p^{(\kappa\,.\,f+2)}$ (mod p).

On putting

$$\omega = \frac{\chi'(\theta)}{p^{(\kappa\,.\,f+2)}} = \frac{\zeta - b + b\theta^{e\kappa} + \sum\limits_{2,\,f+1} b^{p-i} b_1 b_2^{\,i-1} \theta^{e\,[(\kappa\,.\,i)-1]}}{p^{(\kappa\,.\,f+2)}},$$

and noting that $\quad r\kappa + 1 - r\,(\kappa + \kappa^2 + \ldots + \kappa^{f+2}) > \frac{1}{2}$,

it follows, as in a previous argument, that

$$\omega^r = [(\zeta - b)^r + b_1\theta^{er\kappa} + \sum_{2,\,f+1} b^{p-i} b_1 b_2^{\,i-1} \theta^{er[(\kappa\,.\,i)-1]}]/p^{r(\kappa\,.\,f+2)-r} + M\,(p^{\frac{1}{2}}).$$

Accordingly the integer

$$\chi_1(\theta) \equiv (\theta^e - pb)^r + p^r b\theta^{er\kappa} + \sum_{2,\,f+1} p^r b^{p-i} b_1 b_2^{\,i-1} \theta^{er[(\kappa\,.\,i)-1]},$$

which belongs to $[\theta]$, is isobaric with $p^{r(\kappa\,.\,f+2)}$ (mod p) and, if

$$\beta = r\,(\kappa\,.\,f+2),$$

$\theta^s \chi(\theta)^t \chi_1(\theta)^u$ is isobaric with $p^{s\epsilon + t\alpha + u\beta}$.

94. Since when $\qquad c' = er\kappa\,(\kappa\,.\,f)$,

$$\begin{aligned} &e - (s-1)\,c' - (t-1)\,c \\ &\qquad = (r'-s)\,er\kappa\,(\kappa\,.\,f) + (r-t)\,e\,[(\kappa\,.\,f)-1] + er\kappa^{f+1} \end{aligned}$$

which > 0 when $s < r'$, $t < r$, all the $1+q$ numbers

$$\left.\begin{array}{l} 1,\ \dfrac{\theta^e}{p},\ \dfrac{\theta^{e-c}\chi(\theta)}{p^2},\ \dfrac{\theta^{e-2c}\chi(\theta)^2}{p^3},\ \ldots\ \dfrac{\theta^{e-(r-1)c}\chi(\theta)^{r-1}}{p^r} \\[2ex] \dfrac{\theta^{e-c'}\chi_1(\theta)}{p^{r+1}},\ \dfrac{\theta^{e-c'-c}\chi_1(\theta)\,\chi(\theta)}{p^{r+2}},\ \ldots\ \dfrac{\theta^{e-c'-(r-1)c}\chi_1(\theta)\,\chi(\theta)^{r-1}}{p^{2r}} \\[2ex] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex] \dfrac{\theta^{e-(r'-1)c'}\chi_1(\theta)^{r'-1}}{p^{q-r+1}},\ \ldots\ \dfrac{\theta^{e-(r'-1)c'-(r-1)c}\chi_1(\theta)^{r'-1}\chi(\theta)^{r-1}}{p^q} \end{array}\right\} \quad (49)$$

are integral. Each of them becomes fractional, however, when a single factor θ is removed from the numerator. For taking the $(t+1)$th term in the $(s+1)$th row

$$\begin{aligned} e - sc' - tc - 1 + sre(\kappa . f + 2) + te(\kappa . f + 1) \\ = e(rs + t + 1) \\ - (r' - 1 - s)\kappa - (r - 1 - t)\kappa r - \kappa \end{aligned}$$

and the expression in the last line is negative when

$$0 \leqslant s \leqslant r' - 1, \quad 0 \leqslant t \leqslant r - 1.$$

The last $q-1$ integers (49) are also heterobaric with $\theta^y \pmod{p}$ for every integral y. Further, no two of the last q integers (49) are isobaric with each other $(\mathrm{mod}\, p)$, for

$$\begin{aligned} &[e - sc' - tc + sre(\kappa . f + 2) + te(\kappa . f + 1) - e(sr + t + 1)] \\ &- [e - s'c' - t'c + s're(\kappa . f + 2) + t'e(\kappa . f + 1) - e(s'r + t' + 1)] \\ &= \kappa(s - s') + \kappa r'(t - t') \text{ which} \neq 0. \end{aligned}$$

It follows immediately that the $q+1$ integers (49) form the stem of $\mathfrak{o}$ (bas p) except when

$$q = p, \quad f > 0, \quad b^{p-1} \equiv 1 + p \pmod{p^2}.$$

The simple form of the stem when $k \equiv 0 \pmod{h}$ has already been noted in § 92.

Another simple case occurs when $f = 0$, whence

$$e = r', \quad n = qr', \quad c = 0, \quad c' = 1, \quad \chi(\theta) = \theta^e - pb,$$

the stem of $\mathfrak{o}$ (bas p) then being

$$\begin{array}{l} 1, \dfrac{\theta^e}{p}, \dfrac{\theta^e \chi(\theta)}{p^2}, \dfrac{\theta^e \chi(\theta)^2}{p^3}, \ldots \dfrac{\theta^e \chi(\theta)^{r-1}}{p^r}, \\ \dfrac{\theta^{e-1} \chi(\theta)^r}{p^{r+1}}, \dfrac{\theta^{e-1} \chi(\theta)^{r+1}}{p^{r+2}}, \ldots \dfrac{\theta^{e-1} \chi(\theta)^{2r-1}}{p^{2r}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\theta \chi(\theta)^{q-r}}{p^{q-r+1}}, \dfrac{\theta \chi(\theta)^{q-r+1}}{p^{q-r+2}}, \ldots \dfrac{\theta \chi(\theta)^{q-1}}{p^q}. \end{array}$$

95—98. VII. $n = p^{k'}$, $a = p^{mq} b$, $q = p^h$, $dv(m, p) = dv(b, p) = 1$, $b^p - b \not\equiv 0 \pmod{p^2}$, omitting an exception.

95. The cases in which $k' \leqslant h$ and $h = 0$ fall under II and IV and can be disregarded here. All other fields are defined by

$$\theta^e = p^m \zeta, \quad \zeta^q = b, \quad e = p^k, \quad k' = k + h,$$

and, as in VI, it is convenient to put

$$\epsilon = e^{-1}, \quad \mu = m^{-1}, \quad \kappa = q^{-1}, \quad e = q^f r',$$
$$0 \leqslant f \text{ (an integer)}, \quad 1 \leqslant r' < q = rr'.$$

The first $m+1$ terms of the stem of $\mathbf{o}$ (bas p) are

$$1, \frac{\theta^{E(\epsilon\mu)}}{p}, \frac{\theta^{E(2e\mu)}}{p^2}, \dots \frac{\theta^{E(e-e\mu)}}{p^{m-1}}, \frac{\theta^e}{p^m}$$

where $E(z)$ is the integer next greater than (or equal to) z, but since θ^e is isobaric with p^m (mod p) the next term is not generally

$$\theta^{e+E(e\mu)}/p^{m+1}.$$

It can be proved as in § 88 that $\theta^e - p^m b$ is isobaric with $p^{m+\kappa}$ (mod p). When $f=0$ the number $\theta^e - p^m b$ is heterobaric with θ^y for every integral exponent y and the process of § 78 stops at the first stage.

96. More generally when $f \geqslant 1$ the number $\theta^e - p^m b$ is isobaric with $p^{u_1}\theta^{e\kappa v_1}$ provided u_1, v_1 are the unique positive integers such that

$$mq+1 = qu_1 + mv_1, \quad 0 < v_1 < q.$$

If $$\omega_2 = \frac{\theta^e - p^m b + x_2 p^{u_1}\theta^{e\kappa v_1}}{p^{m+\kappa+\sigma}} = \frac{\zeta - b + x_2 p^{u_1-m}\theta^{e\kappa v_1}}{p^{\kappa+\sigma}}$$

it readily follows that

$$\omega_2{}^q = p^{-1-q\sigma}\left[(\zeta-b)^q + px_2\zeta^{v_1}\right] + \text{a complex integer},$$

so long as $1 + q\sigma \leqslant 2$. Now

$$\begin{aligned}(\zeta-b)^q &+ px_2\zeta^{v_1}\\ &= b - b^q - q(\zeta-b)\,b^{q-1} - \sum_{2,\,q-1} {}^qC_i\,(\zeta-b)^i b^{q-i}\\ &\quad + px_2\left[b^{v_1} + v_1(\zeta-b)\,b^{v_1-1} + \sum_{2,\,v_1} {}^{v_1}C_i\,(\zeta-b)^i b^{v_1-i}\right],\end{aligned}$$

so taking

$$x_2 = b_m = \text{least positive residue of } (b^q - b)/pb^{v_1} \pmod p \qquad (50)$$

$(\zeta-b)^q + pb_m\zeta^{v_1}$ is divisible by $p^{1+\kappa}$. It is only divisible by a higher power of p when

$$\frac{b^{q-1}-1}{p}v_1 - \frac{q}{p}b^{q-1} \equiv 0 \pmod p.$$

Noting that $mv_1 \equiv 1$ (mod p), and $b^{q-1} \equiv b^{p-1}$ (mod p^2), this last involves

$$\left.\begin{aligned} &b^{p-1} \equiv 1 + mp \pmod{p^2}\\ \text{and}\qquad &q = p\\ \text{which with}\qquad &f > 0\end{aligned}\right\} \text{.....................(51)}$$

define the exceptional fields. The discussion of these cases is resumed in VIII—X and, as in VI, every field for which $q=2$ is exceptional.

When the conditions (51) are not satisfied $(\zeta-b)^q + pb_m\zeta^{v_1}$ is isobaric with $p^{1+\kappa}$ (mod p) and $\theta^e - p^m b + b_m p^{u_1}\theta^{e\kappa v_1}$ with $p^{m+\kappa+\kappa\kappa}$ so that $\sigma = \kappa^2$.

97. The approximation now continues exactly as in VI. When $f \geqslant 2$ there exist unique positive integers u_2, v_2 making

$$mq^2 + q + 1 = q^2 u_2 + m(v_1 + qv_2), \quad 0 \leqslant v_2 < q,$$

and if

$$a_m \equiv v_1 b_m b^{v_1 - 1} - b^{q-1} q/p \pmod{p},$$

i.e.

$$\left.\begin{aligned} a_m &= v_1 b_m b^{v_1-1} \text{ when } q > p \\ a_m &= v_1 b_m b^{v_1-1} - 1 \text{ when } q = p \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(52),$$

whence $dv(a_m, p) = 1$, the number

$$\theta^e - p^m b + b_m p^{u_1} \theta^{e\kappa v_1} + a_m b_m b^{q-1-v_2} p^{u_2} \theta^{e\kappa(v_2 + \kappa v_1)}$$

is isobaric with $p^{m+\kappa(\kappa \,.\, 2)} \pmod{p}$. Unique positive integers u_i, v_i, with $0 \leqslant v_i < q$, $i \geqslant 1$, also satisfy

$$\left.\begin{aligned} &mq^t + q^{t-1} + q^{t-2} + \ldots + q + 1 \\ &\quad = q^t u_t + m(v_1 + qv_2 + \ldots + q^{t-1} v_t) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(53).$$

The number $p^{m+\kappa(\kappa \,.\, f)}$ is then isobaric $\pmod{p}$ with

$$\begin{aligned} \chi_f(\theta) \equiv \chi(\theta) \equiv \theta^e - p^m b + b_m p^{u_1} \theta^{e\kappa v_1} \\ + \sum_{2,f} a_m{}^{i-1} b_m b^{q-1-v_2-\ldots-v_i} p^{u_i} \theta^{e\kappa^i(v_1 + qv_2 + \ldots + q^{i-1} v_i)}, \end{aligned}$$

whereas $\chi_{f+1}(\theta)$, which does not belong to $[\theta]$ is isobaric with $p^{m+\kappa(\kappa \,.\, f+1)}$. It is readily seen, however, as in § 93, that

$$\begin{aligned} \chi_1(\theta) \equiv (\theta^e - p^m b)^r + b_m p^{ru_1} \theta^{er\kappa v_1} \\ + \sum_{2,f} a_m{}^{i-1} b_m b^{q-1-v_2-\ldots-v_i} p^{ru_i} \theta^{er\kappa^i(v_1 + qv_2 + \ldots + q^{i-1} v_i)}, \end{aligned}$$

a member of the field, is isobaric with $p^{mr + r\kappa(\kappa \,.\, f+1)}$.

The stem of $\mathbf{o}$ (bas p) contains $mq + 1$ terms of which the first and $(m+1)$th are 1 and θ^e/p^m. All its other terms are included in the formula

$$\frac{\theta^x \chi(\theta)^t \chi_1(\theta)^s}{p^{mrs + mt + w + 1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54),$$

for

$$0 \leqslant w < m, \quad 0 \leqslant t < r, \quad 0 \leqslant s < r',$$

provided x is the integer next greater than

$$\frac{e}{m}\left[w + 1 - \frac{sr\kappa(1 - \kappa^{f+2})}{1 - \kappa} - \frac{t\kappa(1 - \kappa^{f+1})}{1 - \kappa}\right].$$

98. The sequence of integers $v_1, v_2, v_3, \ldots,$ defined by (53) is periodic, since it is readily seen that

$$[m(q-1)]^{-1} = 0 \,.\, \dot{v}_L v_{L-1} \ldots v_2 \dot{v}_1$$

as a circulating radix-fraction in scale q.

When $m = 1$ the solution of (53) is

$$u_t = 1, \quad v_1 = v_2 = \ldots = v_t = 1,$$

and this accounts for the comparative simplicity of $\chi(\theta)$ in VI.

There is an obvious modification in the form of $\chi_1(\theta)$, both here and in VI, when $f = 0$, viz.:

$$\chi_1(\theta) = (\theta^e - p^m b)^r + b_m p^{ru_1} \theta^{v_1}.$$

99—103. VIII. p odd, $n = p^{1+k}$, $a = p^{mp} b$, $k > 0$, $dv(m, p) = 1$, $b^{p-1} \equiv 1 + mp \pmod{p^2}$.

In these cases, omitted as exceptional from VI when $m = 1$ and from VII when $m > 1$, we put

$$p^k = e, \quad \theta^e = p^m \zeta, \quad \zeta^p = b, \quad p\pi = m\mu = 1.$$

The first m terms of the stem of $\mathbf{o}$ (bas p) are

$$1, \ \frac{\theta^{E(e\mu)}}{p}, \ \frac{\theta^{E(2e\mu)}}{p^2}, \ \dots \ \frac{\theta^{E(e-e\mu)}}{p^{m-1}}.$$

99. Unique positive integers r_1, s are such that

$$pr_1 + ms = mp + 1$$

and then $\theta^e - p^m b$, $p^{r_1}\theta^{e\pi s}$, $p^{m+\pi}$ are isobaric with each other $(\operatorname{mod} p)$. Putting

$$\omega_2 = \frac{\theta^e - p^m b + x_2 p^{r_1} \theta^{e\pi s}}{p^{m+\pi+\rho}} = \frac{\zeta - b + x_2 p^{r_1 - m} \theta^{e\pi s}}{p^{\pi+\rho}},$$

$$\omega_2^p = \frac{(\zeta - b)^p + x_2 p^{pr_1 - pm} \theta^{es}}{p^{1+p\rho}} + \text{a complex integer} \ \dots\dots(55),$$

so long as $1 + p\rho \leqslant 2$. Now

$$(\zeta - b)^p = b - b^p - p(\zeta - b)\, b^{p-1} - \tfrac{1}{2}p(p-1)(\xi - b)^2 b^{p-2} - \text{etc.},$$

and

$$x_2 p^{pr_1 - pm} \theta^{es} = x_2 p \zeta^s = x_2 p \{b + (\zeta - b)\}^s$$
$$= x_2 p \{b^s + s(\zeta - b)\, b^{s-1} + \tfrac{1}{2} s(s-1)(\zeta - b)^2 b^{s-2} + \dots\},$$

hence, taking $x_2 = mb^{p-s}$, the numerator of (55), being

$$\equiv 2p^{-2p} (\zeta - b)^2 \{-(p-1)\, b^{p-2} + mb^{p-s} s(s-1)\} \pmod{p^{1+3\pi}},$$

is isobaric with $p^{1+2\pi}$ $(\operatorname{mod} p)$. (The argument fails here when $p = 2$.) Hence

$$\omega_2 \equiv \frac{\theta^e - p^m b + mb^{p-s} p^{r_1} \theta^{e\pi s}}{p^{m+\pi+2\pi\pi}}$$

is isobaric with unity $(\operatorname{mod} p)$ and the method of § 78 can be applied again.

100. Next putting $s_1 = 2s$ when $2s < p$ and $s_1 = 2s - p$ when $2s > p$, unique positive integers r_2, s_2' satisfy

$$mp^2 + p + 2 = p^2 r_2 + m(s_1 + ps_2'), \quad 0 \leqslant s_2' < p.$$

It readily follows that

$$\theta^e - p^m b + mb^{p-s} p^{r_1} \theta^{e\pi s} - 2^{p-2} m^2 s b^{3p-2s-s_2'-3} p^{r_2} \theta^{e\pi\pi(s_1 + ps_2')}$$

is isobaric with $p^{m+\pi+2\pi^2+2\pi^3}$ $(\operatorname{mod} p)$.

The approximation is now continued by induction. A unique sequence of positive integers is defined by

$$\left.\begin{aligned} & mp^i + p^{i-1} + 2p^{i-2} + \dots + 2p + 2 \\ & \quad = p^i r_i + m\left[s_1 + ps_2 + \dots + p^{i-2}s_{i-1} + p^{i-1}s_i'\right] \\ & 0 \leqslant s_i' < p, \quad 0 \leqslant s_i < p, \\ & s_i = s_i' + s \text{ when } s_i' + s < p, \\ & s_i = s_i' + s - p \text{ when } s_i' + s \geqslant p \end{aligned}\right\} \dots\dots(56).$$

Putting, when $i \geqslant 4$,

$$\begin{aligned} x_i &= -sb^{p-2-s'_{i-1}}x_2x_{i-1} \\ &= (-s)^{i-2} . 2^{p-2} m^{i-1} b^{(2i-3)p-(i-1)s-s_2'-\dots-s_{i-1}'-2i+4}, \end{aligned}$$

and, when $i \geqslant 2$,

$$w_i = (s_1 + ps_2 + \dots + p^{i-2}s_{i-1} + p^{i-1}s_i')\, e\pi^i$$

the number $\chi(\theta)$, defined by

$$\left.\begin{aligned} \chi(\theta) &= \theta^p - p^m b + mb^{p-s}p^{r_1}\theta^s \text{ when } k = 1 \\ \chi(\theta) &= \theta^e - p^m b + mb^{p-s}p^{r_1}\theta^s + \sum_{3,\, k+1} x_i p^{r_{i-1}}\theta^{w_{i-1}} \end{aligned}\right\} \dots\dots(57),$$

when $k > 1$, is isobaric with $p^{m+\pi+2\pi\pi(\pi . k-1)}$.

101. Noting that

$$(p-2)^{-1} > \pi + 2\pi^2 + \dots + 2\pi^{k+1} > (p-1)^{-1}$$

a unique sequence of positive integers $g(mi+j)$ is defined by

$$0 < i\left[m + \pi + 2\pi^2(1-\pi^k)/(1-\pi)\right] + m\pi^k g(mi+j) - (mi+j) < m\pi^k$$

for $1 \leqslant i \leqslant p-2$, $1 \leqslant j \leqslant m$, and by

$$\begin{aligned} 0 < (p-1)&\left[m + \pi + 2\pi^2(1-\pi^k)/(1-\pi)\right] \\ &+ m\pi^k g(mp-m+j) - (mp-m+j+1) < m\pi^k \end{aligned}$$

for $1 \leqslant j \leqslant m$.

The $mp+2$ integers

$$\left.\begin{aligned} & 1,\ \frac{\theta^{E(e\mu)}}{p},\ \frac{\theta^{E(2e\mu)}}{p^2},\ \dots\ \frac{\theta^{E(e-e\mu)}}{p^{m-1}},\ \frac{\chi(\theta)}{p^m}, \\ & \frac{\theta^{g(m+1)}\chi(\theta)}{p^{m+1}},\ \frac{\theta^{g(m+2)}\chi(\theta)}{p^{m+2}},\ \dots\ \frac{\theta^{g(2m)}\chi(\theta)}{p^{2m}}, \\ & \quad \dots\ \frac{\theta^{g(pm-2m+1)}\chi(\theta)^{p-2}}{p^{pm-2m+1}},\ \frac{\theta^{g(pm-2m+2)}\chi(\theta)^{p-2}}{p^{pm-2m+2}},\ \dots\ \frac{\theta^{g(pm-m)}\chi(\theta)^{p-2}}{p^{pm-m}} \\ & \frac{\chi(\theta)^{p-1}}{p^{pm-m+1}},\ \frac{\theta^{g(pm-m+1)}\chi(\theta)^{p-1}}{p^{pm-m+2}},\ \dots\ \frac{\theta^{g(pm)}\chi(\theta)^{p-1}}{p^{pm+1}} \end{aligned}\right\}$$

$$\dots\dots\dots(58),$$

are all heterobaric (mod p) and form the stem of $\mathfrak{o}$ (bas p).

D, the discriminant, $\equiv 0 \pmod{p^{kep+ep-e+1}}$ exactly.

It is most remarkable that this set of exceptional fields should only exist when $q = p$. An investigation of the prime ideal factors of p in VI, VII and VIII would probably reveal other peculiarities.

102. If M is the least solution of

$$\{\tfrac{1}{2}(p-1)\, m\}^z \equiv 1 \pmod{p},$$

the fraction $2/(p-1)\, m$ can be expanded as a circulating radix-fraction in scale p,

$$0 \,.\, \dot{s}_M s_{M-1} \ldots s_2 \dot{s}_1,$$

thus the sequence of integers s_1, s_2, etc. is periodic.

As a numerical example I take

$$n = 3^3, \quad a = 3^6 \,.\, 5, \quad k = 2, \quad m = 2,$$

which belongs to an exceptional class since

$$5^2 \equiv 1 + 3 \,.\, 2 \pmod{3^2}.$$

$$2/(p-1)\, m = \tfrac{1}{2} = 0 \,.\, \dot{1}$$

in scale 3, so

$$\chi(\theta) \equiv \theta^9 - 3^2 \,.\, 5 + 3\theta^6 + 6\theta^7$$

is isobaric with $3^{71/27}$. The stem of $\mathfrak{o}$ (bas 3) is

$$1,\ \frac{\theta^5}{3},\ \frac{\chi(\theta)}{3^2},\ \frac{\theta^2\chi(\theta)}{3^3},\ \frac{\theta^7\chi(\theta)}{3^4},\ \frac{\chi(\theta)^2}{3^5},\ \frac{\theta^4\chi(\theta)^2}{3^6},\ \frac{\theta^8\chi(\theta)^2}{3^7}.$$

and $D = -3^{73} \,.\, 5^{26}$.

On the other hand, in the field defined by

$$n = 3^3, \quad a = 3^6 \,.\, 2,$$

which follows the normal case VII, the number

$$\chi(\theta) \equiv \theta^9 - 3^2 \,.\, 2 + 6\theta^6 + 18\theta^2$$

is isobaric with $3^{67/27}$. The stem of $\mathfrak{o}$ is

$$1,\ \frac{\theta^5}{3},\ \frac{\chi(\theta)}{3^2},\ \frac{\theta^3\chi(\theta)}{3^3},\ \frac{\theta^7\chi(\theta)}{3^4},\ \frac{\theta\chi(\theta)^2}{3^5},\ \frac{\theta^5\chi(\theta)^2}{3^6} \quad \text{(bas 3)}$$

and $D = -3^{81} \,.\, 2^{26}$.

103. When m is an assigned residue of p the congruence

$$b^{p-1} \equiv 1 + mp \pmod{p^2} \quad \text{......................(59)}$$

has $p-1$ solutions which are incongruent $(\operatorname{mod} p)$. One solution is

$$b \equiv 1 - mp \pmod{p^2}$$

and if $1, y_2, \ldots y_{p-1}$ are the solutions of

$$y^{p-1} - 1 \equiv 0 \pmod{p^2}$$

the complete set of solutions of (59) is

$$1 - mp,\ (1 - mp)\, y_2,\ \ldots\ (1 - mp)\, y_{p-1} \pmod{p^2}.$$

The integer p^2 has $p(p-1)$ residues prime to p: $p-1$ of them satisfy

$$y^{p-1}-1\equiv 0 \pmod{p^2}$$

and appear in the table of II above. Each of the remaining $(p-1)^2$ residues is a solution of (59) for

$$m=1,\ 2,\ \ldots \text{ or } p-1.$$

The solutions when $p=3,\ 5,\ 7$ are appended.

$$p=3:\begin{cases} m\equiv 1 \pmod 3, & b\equiv 7,\ 2 \pmod 9),\\ m\equiv 2 \pmod 3, & b\equiv 4,\ 5 \pmod 9).\end{cases}$$

$$p=5:\begin{cases} m\equiv 1 \pmod 5, & b\equiv 21,\ 22,\ 3,\ 4 \pmod{25},\\ m\equiv 2 \pmod 5, & b\equiv 16,\ 12,\ 13,\ 9 \pmod{25},\\ m\equiv 3 \pmod 5, & b\equiv 11,\ 2,\ 23,\ 14 \pmod{25},\\ m\equiv 4 \pmod 5, & b\equiv 6,\ 17,\ 8,\ 19 \pmod{25}.\end{cases}$$

$$p=7:\begin{cases} m\equiv 1 \pmod 7, & b\equiv 43,\ 16,\ 10,\ 39,\ 33,\ 6 \pmod{49},\\ m\equiv 2 \pmod 7, & b\equiv 36,\ 2,\ 38,\ 11,\ 47,\ 13 \pmod{49},\\ m\equiv 3 \pmod 7, & b\equiv 29,\ 37,\ 17,\ 32,\ 12,\ 20 \pmod{49},\\ m\equiv 4 \pmod 7, & b\equiv 22,\ 23,\ 45,\ 4,\ 26,\ 27 \pmod{49},\\ m\equiv 5 \pmod 7, & b\equiv 15,\ 9,\ 24,\ 25,\ 40,\ 34 \pmod{49},\\ m\equiv 6 \pmod 7, & b\equiv 8,\ 44,\ 3,\ 46,\ 5,\ 41 \pmod{49}.\end{cases}$$

104. IX. $p=2$, $n=2^{1+k}$, $k\geqslant 1$, $a=2^{2m}b$, $m\equiv 1 \pmod 2$, $b\equiv 3 \pmod 8$. In these exceptional cases, omitted from VI and VII, put

$$q=2^{k-1}=\tfrac{1}{4}n,\quad 2\pi=m\mu=1.$$

Every two integers of the field, of rank lower than $2q$, are heterobaric (mod 2) whereas

$$\theta^{2q}+2^m b,\quad 2^{(m+1)\pi}\theta^q,\quad 2^{m+\pi}$$

are isobaric with each other. The number

$$\omega\equiv(\theta^{2q}+2^{(m+1)\pi}\theta^q-2^m)/2^{m+1}$$

is a root of

$$\omega^4+2\omega^3-\tfrac{1}{2}(b-3)\,\omega^2-\tfrac{1}{2}(3b-1)\,\omega+\tfrac{1}{16}(b^2-14b+1)=0$$

and so isobaric with $\sqrt[4]{2}$ since

$$b^2-14b+1\equiv 32 \pmod{64}.$$

(The argument fails here when $b\equiv 7 \pmod 8$ since then

$$b^2-14b+1\equiv 16 \pmod{32}.)$$

It can now be readily seen that

$$\theta^{2q}-2^m+2^{(m+1)\pi}\theta^q+2^{(m+5)\pi\pi}\theta^{3q\pi}$$

is isobaric with $2^{m+1+\pi-\pi^3}$ (mod 2). Also if

$$2^i(m+1)+2^{i-1}-1=2^i u_i+mv_i$$

with $0 \leqslant v_i < 2^i$ when $i = 2, 3, \dots k$, it can be proved as in VI that

$$\chi(\theta) \equiv \theta^{2q} - 2^m + 2^{(m+1)\pi}\theta^q + \sum_{2,k} 2^{u_i}\theta^{q\pi^i v_i}$$

is isobaric with $2^{m+1+\pi-1/n}$. The field $[\theta]$ contains no integer of rank $2q$ (or lower) isobaric with $\chi(\theta)$ (mod 2). Putting

$$g(i) = E\left[\frac{2q}{m}\left(i - \frac{1}{2} + \frac{1}{2^{k+1}}\right)\right], \quad 1 \leqslant i \leqslant m,$$

the $2m+2$ integers

$$1, \frac{\theta^{E(2q\mu)}}{2}, \frac{\theta^{E(4q\mu)}}{2^2}, \dots \frac{\theta^{E(2q-2q\mu)}}{2^{m-1}},$$

$$\frac{\chi(\theta)}{2^m}, \frac{\chi(\theta)}{2^{m+1}}, \frac{\theta^{g(1)}\chi(\theta)}{2^{m+2}}, \dots \frac{\theta^{g(m)}\chi(\theta)}{2^{2m+1}}$$

are heterobaric (mod 2) and form the stem of $\mathfrak{o}$ (bas 2). When $m = 1$ or $m > q$, $g(m) \geqslant 2q$ and the last term in the stem as above is omitted.

D, the discriminant, $\equiv 0$ (mod $2^{q(4k+1+m)}$) exactly.

In the simplest case when $m = 1$, $n = 4$ the stem of $\mathfrak{o}$ reduces to

$$1, \frac{\theta^2}{2}, \frac{\theta^2 + 2\theta + 2}{2^2} \text{ (bas 2)},$$

and, when $m = 1$, $n = 4q > 4$, it becomes

$$1; \frac{\theta^{2q}}{2}, \frac{\theta^{2q} + 2\theta^q + 2}{2^2}, \frac{\theta^{q+1}(\theta^{2q} + 2\theta^q + 2)}{2^3} \text{ (bas 2)}.$$

105. X. $p = 2$, $n = 2^{1+k}$, $k \geqslant 1$, $a = 2^{2m}b$, $m \equiv 1$ (mod 2), $b \equiv 7$ (mod 8); put also $q = 2^{k-1}$, $2\pi = m\mu = 1$.

The field $[\theta]$ can be defined by

$$\theta^q = \alpha, \quad \alpha^4 = 2^{2m}b,$$

and will be considered as a super-field of $[\alpha]$. Since the third dissection fails to separate the prime ideal factors of 2 in $[\alpha]$ when represented by $\alpha^4 - 2^{2m}b = 0$ we apply the transformation

$$2^{m+1}\beta = \alpha^2 + 2^{(m+1)\pi}\alpha + 2^m,$$

giving

$$B(\beta) \equiv \beta^4 - 2\beta^3 - \tfrac{1}{2}(b-3)\beta^2 - \tfrac{1}{2}(b+1)\beta + \tfrac{1}{16}(b+1)^2 = 0.$$

The only irreducible divisor of $B(z)$ (mod 2) is z.

First, when $b \equiv 7$ (mod 16)

$$B(z) \equiv z^4 + 2z^2 + 4 \text{ (modd } \{z, \sqrt{2}\}^5),$$

which is irreducible: hence

$$\mathfrak{o} \equiv (1, \beta, \tfrac{1}{2}\beta^2, \tfrac{1}{2}\beta^3) \text{ (bas 2)}$$

$$\equiv \left(1, \frac{\alpha}{2^{(m-1)\pi}}, \frac{\alpha^2 + 2^{(m+1)\pi}\alpha + 2^m}{2^{m+1}}, \frac{\alpha^3 + 2^{(m+1)\pi}\alpha^2 + 2^m\alpha}{2^{(3m+3)\pi}}\right) \text{ (bas 2)}.$$

Also $(2) = \mathfrak{p}^2$ where

$$\mathfrak{p} \equiv (2,\ \beta,\ \beta^2,\ \tfrac{1}{2}\beta^3)\ (\text{bas } 2)$$

$$\equiv \left(2,\ \frac{\alpha}{2^{(m-1)\pi}},\ \frac{\alpha^2 + 2^{(m+1)\pi}\alpha + 2^m}{2^{m+1}},\ \frac{\alpha^3 + 2^{(m+1)\pi}\alpha^2 + 2^m\alpha}{2^{(3m+1)\pi}}\right)\ (\text{bas } 2).$$

Secondly when $b \equiv 15 \pmod{16}$, $b + 1 \equiv 2^t \pmod{2^{t+1}}$, $t \geqslant 4$, and the diagram contains two edges of slants $\frac{1}{2}$, $\frac{1}{2}(2t-5)$.

Hence

$$\mathfrak{o} \equiv \left(1,\ \beta,\ \frac{\beta^2}{2},\ \frac{\beta^3 - 2\beta^2 + \frac{1}{2}(3-b)\beta}{2^{t-1}}\right)\ (\text{bas } 2)$$

$$\equiv \left(1,\ \frac{\alpha}{2^{(m-1)\pi}},\ \frac{\alpha^2 + 2^{(m+1)\pi}\alpha + 2^m}{2^{m+1}},\ \frac{\alpha^3 + 2^{(m+1)\pi}\alpha^2 + 2^m\alpha}{2^{(3m+3)\pi}}\right)\ (\text{bas } 2).$$

Further $(2) = \mathfrak{p}_1{}^2\mathfrak{p}_2{}^2$ where

$$\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2 \equiv \left(2,\ \beta,\ \beta^2,\ \frac{\beta^3 - 2\beta^2 + \frac{1}{2}(3-b)\beta}{2^{t-1}}\right)\ (\text{bas } 2)$$

$$\equiv \left(2,\ \frac{\alpha}{2^{(m-1)\pi}},\ \frac{\alpha^2 + 2^{(m+1)\pi}\alpha + 2^m}{2^{m+1}},\ \frac{\alpha^3 + 2^{(m+1)\pi}\alpha^2 + 2^m\alpha}{2^{(3m+1)\pi}}\right)\ (\text{bas } 2).$$

Each prime ideal factor of 2 divides α just m times, and each $\mathfrak{p}$-diagram for $\theta^q = \alpha$ contains but one edge joining $(q, 0)$, $(0, m)$. Hence, as previously in IV,

$$\mathfrak{O} = \left(\mathfrak{o},\ \frac{\theta^{E(q\mu)}\mathfrak{p}}{2},\ \frac{\theta^{E(2q\mu)}\mathfrak{p}^2}{2^2},\ \dots\ \frac{\theta^{E(q-q\mu)}\mathfrak{p}^{m-1}}{2^{m-1}}\right)\ (\text{bas } 2).$$

$\mathfrak{D}$, the relative discriminant of $[\theta]$ with respect to $[\alpha]$, is exactly divisible by $\mathfrak{p}^{2q(k-1)+q-1}$ whereas

$$D = d^q N_\alpha(\mathfrak{D}) \equiv 0 \pmod{2^{4qk+2q-2}} \text{ exactly.}$$

106. XI. $\qquad n = lp^k, \quad a = p^{mq}b, \quad q = p^h, \quad k > h,$

$$b^p - b \not\equiv 0 \pmod{p^2}, \quad dv(b, p) = dv(l, p) = dv(m, p) = 1.$$

This reduces to IV when $q = 1$: in other cases the field can be defined by

$$\theta^l = \alpha, \quad \alpha^{p^k} = p^{mp^h}b$$

and will be considered as a super-field of that discussed in VI, VII, VIII, IX or X. In each of these cases it has been shewn that the $(t+1)$th term of the stem of $[\alpha]$ (bas p) is

$$\alpha^x\chi(\alpha)^y\chi_1(\alpha)^z/p^t$$

where $\chi(\alpha)$, $\chi_1(\alpha)$, of ranks r, s are respectively isobaric with p^g, $p^f \pmod{p}$ and x, y, z are positive integers (or zero) so chosen that

$$x + yr + zs$$

is least, subject to the condition

$$xmp^{h-k} + yg + zf \geqslant t.$$

The $(t+1)$th term of the stem of $[\theta]$ (bas p) is

$$\theta^{x'}\chi(\theta)^{y'}\chi_1(\theta)^{z'}$$

where x', y', z' are so chosen that

$$x' + y'lr + z'ls$$

is least, subject to $\quad x'mp^{h-k}/l + y'g + z'f \geqslant t.$

107. XII. $\quad n = lp^k, \quad a = p^{mq}b, \quad q = p^h, \quad k \leqslant h,$
$dv(b, p) = dv(l, p) = dv(m, p) = 1;$

put also $p^k = e$, $q = er$.

This field can be defined by

$$\theta^l = p^{mr}\alpha, \quad \alpha^e = b,$$

and the prime ideal factors of p in $[\alpha]$ have already been separated in II. A prime ideal factor $\mathfrak{p}$ of p divides p just

$$p^g(p-1)^c \text{ times}, \quad 0 \leqslant g \leqslant e, \quad c = 0 \text{ or } 1;$$

it therefore divides p^{mr} or θ^l just $p^{h+g-k}(p-1)^c m$ times. Hence, since l is prime to p,

$$\mathfrak{O} \equiv \left(\mathfrak{o}, \frac{\mathfrak{o}\theta^{E(l\mu)}}{\mathfrak{p}}, \frac{\mathfrak{o}\theta^{E(2l\mu)}}{\mathfrak{p}^2}, \dots \frac{\mathfrak{o}\theta^{E(l-l\mu)}}{\mathfrak{p}^{v-1}}\right) (\text{bas } \mathfrak{p})$$

where $\quad v = \mu^{-1} = p^{h+g-k}(p-1)m$

and $\mathfrak{o}$ is the integral basis of $[\alpha]$. The partial basis of $[\theta]$ (bas p) is obtained by combining, as in § 71, the partial bases for the different prime ideal factors of p.

108. XIII—XXIII. $\quad n = lp^{k'}, \quad a = p^{mq}b, \quad q = p^h, \quad k' > h \geqslant 1,$
$b^p - b \equiv 0 \pmod{p^2}, \quad dv(b, p) = dv(l, p) = dv(m, p) = 1.$

Put also $\quad k' = k + h, \quad e = p^k \geqslant p,$

then $\quad (\theta^{lp^k}/p^m)^{p^h} = b,$

and the field can be defined by

$$\theta^{le} = p^m\alpha, \quad \alpha^q = b.$$

The stem of $\mathfrak{O}$ (bas p) for the relative-field is a combination of its stems for the several prime ideal factors of p in $[\alpha]$—and these factors have been separated in II. Eleven cases now need separate discussion, $\mathfrak{p}$ dividing p just $p^g(p-1)$ or p^g times.

XIII. $\mathfrak{p} = \mathfrak{p}_1, \dots \mathfrak{p}_{j-1}$ of (34), $\mathfrak{p}_1, \dots \mathfrak{p}_{j-2}$ of (35) or $\mathfrak{p}_1, \dots \mathfrak{p}_{h-2}$ of (36), $g > e \geqslant p$.

XIV. $\mathfrak{p} = \mathfrak{p}_1, \dots \mathfrak{p}_{j-1}$ of (34), $\mathfrak{p}_1, \dots \mathfrak{p}_{j-3}$ of (35), $\mathfrak{p}_1, \dots \mathfrak{p}_{h-2}$ of (36), also $\mathfrak{p}_{j-2}$ of (35) when $j < h+1$ and $\mathfrak{p}_{h-1}$ of (36) when $p > 2$; $e = p^k \geqslant g \geqslant p$.

XV. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-2}$ of (35) when $j=h+1$; $g=2$.
XVI. $p=2$, $\mathfrak{p}=\mathfrak{p}_{h-1}$ of (36); $g=2$.
XVII. p odd, $\mathfrak{p}=\mathfrak{p}_j$ of (34), $g>e\geqslant p$.
XVIII. p odd, $\mathfrak{p}=\mathfrak{p}_j$ of (34), $e\geqslant g\geqslant p$.
XIX. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $g>e\geqslant 2$.
XX. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $e\geqslant g>2$.
XXI. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $e\geqslant g=2$.
XXII. $\mathfrak{p}=\mathfrak{p}_h$ of (36).
XXIII. $\mathfrak{p}=\mathfrak{p}_{h+1}$ of (36).

109. XIII. $\mathfrak{p}=\mathfrak{p}_1, \ldots \mathfrak{p}_{j-1}$ of (34), $\mathfrak{p}_1, \ldots \mathfrak{p}_{j-2}$ of (35), $\mathfrak{p}_1, \ldots \mathfrak{p}_{h-2}$ of (36), $g>e\geqslant p$; also $e\epsilon=1$, $t=\tau^{-1}=mg\,(p-1)\,\epsilon$.

Since θ^{le} is isobaric with $\mathfrak{p}^{mg(p-1)}$ (mod $\mathfrak{p}$) and θ^l with $\mathfrak{p}^{mg\epsilon(p-1)}$, the first t terms of the stem of $\boldsymbol{\Theta}$ (bas $\mathfrak{p}$) are as tabulated in (60) below. Again since

$$\eta=a^{g\epsilon\,(p-1)}+b^{g\epsilon}a^{g\epsilon\,(p-2)}+\ldots+b^{g\epsilon\,(p-1)}\equiv 0 \pmod{\mathfrak{p}^{g\epsilon\,(p-1)}}$$

exactly, after § 83 above, the integer

$$\omega=\theta^l-A\eta^m$$

must, as in § 78, be tested for divisibility by $\mathfrak{p}^{mg\epsilon\,(p-1)+\sigma}$ with $\sigma>0$. Forming the equation of degree e in $[a]$ satisfied by ω,

$$\pm N_a(\omega)=A^e\eta^{me}-p^m a,$$

and nothing is lost in assuming A to be rational since $\mathfrak{p}$ is of the first degree. From § 83 the number

$$\eta_g(a)=a^{g(p-1)}+b^g a^{g(p-2)}+\ldots+b^{g-1}\equiv 0 \pmod{\mathfrak{p}^{g(p-1)+2}}$$

at least, also identically

$$\eta\,(z)^e=\eta_g(z)+b^{g\,(p-1)}(p^e-p)+A_1(z-b)+A_2(z-b)^2+\text{etc.},$$

where A_1, A_2 etc. are rational multiples of p. Differentiating this identity and putting $z=b$ we have

$$A_1\equiv 0 \pmod{p^2},$$

whence

$$\eta\,(a)^e\equiv -p \pmod{\mathfrak{p}^{g(p-1)+2}}.$$

Thus

$$A^e\eta\,(a)^{me}-p^m a$$

$$\equiv p^m\,[b-a+(-)^m A-b] \pmod{\mathfrak{p}^{mg(p-1)+2}},$$

so, taking $A=(-)^m b$,

$$N_a\,[\theta^l-(-)^m b\eta\,(a)^m]\equiv 0 \pmod{\mathfrak{p}^{mg(p-1)+1}} \text{ exactly,}$$

and for no integer A of $[a]$ is

$$N_a(\theta^l-A\eta\,(a)^m)\equiv 0 \pmod{\mathfrak{p}^{mg(p-1)+2}}.$$

It is now evident that

$$\phi \equiv \theta^{l} - (-)^{m} b\eta(\alpha)^{m} \text{ and } \mathfrak{p}^{t+\epsilon}$$

are isobaric (mod $\mathfrak{p}$), so putting

$$c = E[mg(p-1)/l] = E(te/l),$$

every member of the following tableau is integral:

$$\left.\begin{array}{l}
\mathfrak{o},\ \dfrac{\mathfrak{o}\theta^{E(l\tau)}}{\mathfrak{p}},\ \dfrac{\mathfrak{o}\theta^{E(2l\tau)}}{\mathfrak{p}^{2}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(l-l\tau)}}{\mathfrak{p}^{t-1}}, \\
\dfrac{\mathfrak{o}\phi}{\mathfrak{p}^{t}},\ \dfrac{\mathfrak{o}\theta^{E[l\tau(1-\epsilon)]}\phi}{\mathfrak{p}^{t+1}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(l-l\tau-l\tau\epsilon)}\phi}{\mathfrak{p}^{2t-1}}, \\
\qquad\qquad \ldots\ \dfrac{\mathfrak{o}\phi^{c-1}}{\mathfrak{p}^{c(t-1)}},\ \dfrac{\mathfrak{o}\theta^{E[l\tau(1+\epsilon-c\epsilon)]}\phi^{c-1}}{\mathfrak{p}^{c+1(t-1)}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(l-l\tau+l\tau\epsilon-cl\tau\epsilon)}\phi^{c-1}}{\mathfrak{p}^{ct-1}}, \\
\dfrac{\mathfrak{o}\phi^{c}}{\mathfrak{p}^{ct}},\ \dfrac{\mathfrak{o}\theta^{E[l\tau(1-c\epsilon)]}\phi^{c}}{\mathfrak{p}^{ct+1}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(l-l\tau-cl\tau\epsilon)}\phi^{c}}{\mathfrak{p}^{(c+1)t-1}},\ \dfrac{\mathfrak{o}\theta^{E(l-c\epsilon)}\phi^{c}}{\mathfrak{p}^{(c+1)t}}, \\
\qquad\qquad \ldots\ \dfrac{\mathfrak{o}\theta^{E(l\tau\epsilon)}\phi^{e-1}}{\mathfrak{p}^{(e-1)t+1}},\ \dfrac{\mathfrak{o}\theta^{E(l\tau+l\tau\epsilon)}\phi^{e-1}}{\mathfrak{p}^{(e-1)t+2}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(l-l\tau+l\tau\epsilon)}\phi^{e-1}}{\mathfrak{p}^{et}}
\end{array}\right\} \qquad \ldots\ldots\ldots(60).$$

No two of these ideals are isobaric (mod $\mathfrak{p}$) unless they lie in the same row: an obvious extension of the argument in case IV shews at once that the modulus (60) is the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$). When

$$mg(p-1) > l(e-1)$$

the rank of one or more terms in the last row exceeds le and such terms are omitted from the stem.

110. XIV. $\mathfrak{p} = \mathfrak{p}_1, \ldots \mathfrak{p}_{j-1}$ of (34), $\mathfrak{p}_1, \ldots \mathfrak{p}_{j-2}$ of (35), $\mathfrak{p}_1, \ldots \mathfrak{p}_{h-1}$ of (36) except when $g = 2$, $e = p^k \geqslant g \geqslant p$; also

$$e = g^f r',\quad 1 \leqslant r' < g = rr',\quad f \geqslant 1,\quad \gamma = g^{-1},\quad \mu^{-1} = m(p-1).$$

The first $m(p-1)$ terms of the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) are

$$\mathfrak{o},\quad \mathfrak{o}\theta^{E(sle\gamma\mu)}/\mathfrak{p}^{s},\quad 1 \leqslant s \leqslant m(p-1)-1.$$

Next if

$$\omega_1 = \theta^{le\gamma} - A_0(\alpha - b)^{m(p-1)},$$

$$\omega_1^{g} + \sum_{1,\,g-1} {}^{g}C_i A_0^{i}(\alpha - b)^{m(p-1)i}\omega_1^{g-i} + A_0^{g}(\alpha - b)^{mg(p-1)} - p^{m}\alpha = 0.$$

Since ${}^{q}C_g/{}^{q}C_{pg} \equiv p \pmod{p^2}$ the diagram of squares shews that

$$(\alpha - b)^{(p-1)g} \equiv -p \ (\text{mod } \mathfrak{p}^{g(p-1)+2}).$$

(The argument fails here when $g = 2$, whence the omissions in the heading.) Hence

$$(\alpha - b)^{mg(p-1)} \equiv (-p)^{m} \ (\text{mod } \mathfrak{p}^{mg(p-1)+2})$$

and

$$N_{\mathfrak{a}}(\theta^{le\gamma} - A_0(\alpha - b)^{m(p-1)}) \equiv p^{m}(-\alpha + (-)^{m}A_0^{g}) \ (\text{mod } \mathfrak{p}^{mg(p-1)+2}).$$

So taking $A_0 = (-)^m b$ the integer

$$\omega_1 \equiv \theta^{le\gamma} - (-)^m b\,(a-b)^{m(p-1)}$$

is isobaric with $\mathfrak{p}^{m(p-1)+\gamma}$ (mod $\mathfrak{p}$).

An approximation similar to that in VI now shews that, if u_i, v_i, w_i are the unique integers $\geqslant 0$ defined in succession by

$$\begin{aligned} g^i u_i &+ m\,(p-1)\,(v_1 + g v_2 + \ldots + g^{i-1} v_i) \\ &= g^i u_i + m\,(p-1)\,w_i \\ &= m\,(p-1)\,g^i + g^{i-1} + \ldots + 1, \quad 1 \leqslant i \leqslant f-1, \end{aligned}$$

and
$$A_i = (-)^m b^{p-i}\,\{(-)^m b^{p-2}\}^{v_1+\ldots+v_i},$$

then the integer

$$\begin{aligned} \phi\,(\theta) \equiv \theta^{le\gamma} - (-)^m b\,(a-b)^{m(p-1)} - A_1\,(a-b)^{u_1}\theta^{le\gamma\gamma v_1} - \ldots \\ \ldots - A_{f-1}\,(a-b)^{u_{f-1}}\theta^{lr' w_{f-1}} \end{aligned}$$

is isobaric with $\mathfrak{p}^{m(p-1)+\gamma+\gamma\gamma+\ldots+r'\epsilon}$ (mod $\mathfrak{p}$). When $r'=1$, the stem of $\mathfrak{D}$ (bas $\mathfrak{p}$) contains $mg\,(p-1)$, or fewer, ideals, the $[1+tm\,(p-1)]$th of them being $\mathfrak{o}\phi\,(\theta)^t/\mathfrak{p}^{tm(p-1)}$ when

$$(t-1)\,[m\,(p-1) + \gamma + \gamma^2 + \ldots + \gamma^f] + m\,(p-1)\,(le\gamma - 1)/le\gamma < tm\,(p-1),$$

and otherwise $\mathfrak{o}\theta^{x_t}\phi\,(\theta)^{t-1}/\mathfrak{p}^{tm(p-1)}$ where x_t is the unique positive integer such that

$$\begin{aligned} 1 - (t-1)\,m\,(p-1) > x_t - tle\gamma + (t-1)\,(\gamma + \gamma^2 + \ldots + \gamma^f) \\ > -(t-1)\,m\,(p-1). \end{aligned}$$

All the other terms are included in the scheme

$$\frac{\mathfrak{o}\theta^{x_{ts}}\phi\,(\theta)^t}{\mathfrak{p}^{tm(p-1)+s}}, \quad 0 \leqslant t \leqslant g-1, \quad 1 \leqslant s \leqslant m\,(p-1)-1$$

provided x_{ts} is the unique integer satisfying

$$m\,(p-1) > m\,(p-1)\,x_{ts} - sle\gamma + tle\gamma\,(\gamma + \gamma^2 + \ldots + \gamma^f) \geqslant 0.$$

In the more general case, when $r' > 1$, the integer

$$\begin{aligned} \phi_1\,(\theta) \equiv \theta^{ler\gamma} - A_0\,(a-b)^{mr(p-1)} - A_1\,(a-b)^{ru_1}\theta^{le\gamma\gamma r v_1} - \ldots \\ \ldots - A_{f-1}\,(a-b)^{ru_{f-1}}\theta^{lgw_{f-1}} - A_f\,(a-b)^{u_f}\theta^{lw_f} \end{aligned}$$

is isobaric with $\mathfrak{p}^{mr(p-1)+r\gamma+r\gamma\gamma+\ldots+\epsilon}$ (mod $\mathfrak{p}$). The $(1+s)$th term of the stem of $\mathfrak{D}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o}\theta^x\phi\,(\theta)^y\phi_1\,(\theta)^z/\mathfrak{p}^s$$

where x, y, z are the unique set of integers $\geqslant 0$ making

$$x + yle\gamma + zle\gamma r$$

least, subject to the restriction

$$\begin{aligned} xm\,(p-1)/le\gamma + y\,\{m\,(p-1) + \gamma + \gamma^2 + \ldots + \gamma^f\} \\ + zr\,\{m\,(p-1) + \gamma + \gamma^2 + \ldots + \gamma^{f+1}\} \geqslant s. \end{aligned}$$

The sequence of integers $v_1, v_2, \dots$ is periodic as in previous cases. Further, $A_0 = A_1 = \dots = A_f = 1$ whenever $p = 2$.

111. XV. $p = 2$, $\mathfrak{p} = \mathfrak{p}_{j-2}$ of (35), $g = 2$, $j = h + 1$, $q = 2^h \geqslant 4$; also

$$e = 2^k,\ e\epsilon = 2\gamma = m\mu = 1.$$

Since $\theta^{le\gamma} \equiv 0 \pmod{\mathfrak{p}^m}$ exactly, we put

$$\omega_1 = [\theta^{le\gamma} - 2^{(m-1)\gamma}(\alpha - 1)]/2^{(m-1)\gamma},$$

whence

$$\omega_1^2 + 2(\alpha - 1)\omega_1 + (\alpha - 1)^2 - 2\alpha = 0 \dots\dots\dots\dots\dots (61).$$

Now

$$\begin{aligned} N[(\alpha-1)^2 - 2\alpha] &= N(\alpha - 2 - \sqrt{3}) \,.\, N(\alpha - 2 + \sqrt{3}) \\ &= [(2+\sqrt{3})^q - b]\,[(2-\sqrt{3})^q - b] \\ &= 1 + b^2 - b\,[(2+\sqrt{3})^q + (2-\sqrt{3})^q], \end{aligned}$$

and since $(2+\sqrt{3})^q + (2-\sqrt{3})^q \equiv 3 \,.\, 2^{2h+2} + 2 \pmod{2^{2h+4}}$,

$$N[(\alpha-1)^2 - 2\alpha] \equiv (b-1)^2 - 3 \,.\, 2^{2h+2} b \pmod{2^{2h+4}} \equiv 0 \pmod{2^{2h+3}} \text{ exactly.}$$

The prime ideal factors $\mathfrak{p}_1, \dots \mathfrak{p}_{j-3}$ of 2 in $[\alpha]$ are of the first degree and each divides $(\alpha - 1)^2 - 2\alpha$ twice, whereas $\mathfrak{p}_{j-1}$, of the second degree, divides this integer once only. So if

$$(\alpha - 1)^2 - 2\alpha \equiv 0 \pmod{\mathfrak{p}^x} \text{ exactly},$$

$$2h + 3 = 2(j - 3) + x + 2,$$

i.e.

$$x = 5,$$

whence, from (61), since $2(\alpha - 1) \equiv 0 \pmod{\mathfrak{p}^3}$,

$$\theta^{le\gamma} - 2^{(m-1)\gamma}(\alpha - 1), \text{ or } 2^{(m-1)\gamma}\omega_1,$$

is isobaric with $\mathfrak{p}^{m+1+\gamma} \pmod{\mathfrak{p}}$.

Next using the sequence of positive integers u_i, w_i such that

$$2^i u_i + m w_i = 2^i(m + 2) - 1, \quad 1 \leqslant i \leqslant k - 1,$$

it follows without difficulty that

$$\phi(\theta) \equiv \theta^{le\gamma} - 2^{(m-1)\gamma}(\alpha - 1) + \sum_{1,\, k-1} (\theta - 1)^{u_i} \alpha^{le\gamma^{i+1} w_i}$$

is isobaric with $\mathfrak{p}^{m+2-\epsilon} \pmod{\mathfrak{p}}$. Hence as in former cases the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o},\ \frac{\mathfrak{o}\theta^{E(le\gamma\mu)}}{\mathfrak{p}},\ \frac{\mathfrak{o}\theta^{E(2le\gamma\mu)}}{\mathfrak{p}^2},\ \dots\ \frac{\mathfrak{o}\theta^{E(le\gamma - le\gamma\mu)}}{\mathfrak{p}^{m-1}},$$

$$\frac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^m},\ \frac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^{m+1}},$$

$$\frac{\mathfrak{o}\theta^{E(l\gamma\mu)}\phi(\theta)}{\mathfrak{p}^{m+2}},\ \frac{\mathfrak{o}\theta^{E(l\gamma\mu + le\gamma\mu)}\phi(\theta)}{\mathfrak{p}^{m+3}},\ \dots\ \frac{\mathfrak{o}\theta^{E(le\gamma - le\gamma\mu + l\gamma\mu)}\phi(\theta)}{\mathfrak{p}^{2m+1}},$$

the last member being omitted when

$$2m > l(e - 1),$$

for then

$$E(le\gamma - le\gamma\mu + l\gamma\mu) > le\gamma - 1.$$

112. XVI. $p=2$, $\mathfrak{p}=\mathfrak{p}_{h-1}$ of (36), $g=2$, $j>h+1$, $q=2^h \geqslant 4$; also

$$2\gamma = m\mu = 1.$$

Here $\theta^{le\gamma} \equiv 0 \pmod{\mathfrak{p}^m}$ exactly, and if

$$\phi(\theta) = \theta^{le\gamma} - 2^{(m-1)\gamma}(\alpha+1)$$

then

$$\phi(\theta)^2 + 2^{(m+1)\gamma}(\alpha+1)\,\phi(\theta) + 2^{m-1}(\alpha^2+1) = 0 \quad \ldots\ldots(62).$$

Also

$$N(\alpha^2+1) = N(\alpha+i)\,N(\alpha-i) = (b-1)^2$$

which $\equiv 0 \pmod{2^{2j}}$ exactly. The prime ideal factors $\mathfrak{p}_1, \ldots \mathfrak{p}_{h-2}, \mathfrak{p}_h, \mathfrak{p}_{h+1}$ of 2 in $[\alpha]$ are all of the first degree and the first $h-2$ of them divide α^2+1, which $=(\alpha+1)^2-2\alpha$, just twice, the other two once only. Hence if

$$\alpha^2+1 \equiv 0 \pmod{\mathfrak{p}^x} \text{ exactly},$$

$$2(h-2)+x+2 = 2j,$$

i.e.

$$x = 2(j-h+1) \geqslant 6,$$

and, since

$$2^{(m+1)\gamma}(\alpha+1) \equiv 0 \pmod{\mathfrak{p}^{m+2}},$$

it follows from (62) that $\phi(\theta)$ is divisible by $\mathfrak{p}^{m+2}$ though not necessarily isobaric with $\mathfrak{p}^{m+2} \pmod{\mathfrak{p}}$.

Next putting

$$\omega = \theta^{le\gamma} - f(\alpha),$$

so that

$$\omega^2 + 2f(\alpha)\,\omega + f(\alpha)^2 - 2^m\alpha = 0,$$

it is easily seen that no integer $f(\alpha)$ of $[\alpha]$ simultaneously satisfies

$$\left.\begin{aligned} 2f(\alpha) &\equiv 0 \pmod{\mathfrak{p}^{m+3}} \\ f(\alpha)^2 - 2^m\alpha &\equiv 0 \pmod{\mathfrak{p}^{2m+5}} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(63),$$

which are the necessary conditions for divisibility of $\theta^{le\gamma}-f(\alpha)$ by $\mathfrak{p}^{m+2+\sigma}$ with $\sigma>0$.

All members of the compound modulus

$$\left.\begin{array}{l} \mathfrak{o},\ \dfrac{\mathfrak{o}\theta^{E(le\gamma\mu)}}{\mathfrak{p}},\ \dfrac{\mathfrak{o}\theta^{E(2le\gamma\mu)}}{\mathfrak{p}^2},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(le\gamma-le\gamma\mu)}}{\mathfrak{p}^{m-1}}, \\ \dfrac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^m},\ \dfrac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^{m+1}}, \\ \dfrac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^{m+2}},\ \dfrac{\mathfrak{o}\theta^{E(le\gamma\mu)}\,\phi(\theta)}{\mathfrak{p}^{m+3}},\ \ldots\ \dfrac{\mathfrak{o}\theta^{E(le\gamma-le\gamma\mu)}\,\phi(\theta)}{\mathfrak{p}^{2m+1}}, \end{array}\right\} \ldots\ldots(64)$$

are now seen to be integral. The s-th ideal in the first row is isobaric $\pmod{\mathfrak{p}}$ with the s-th ideal in the third row but, after (63), no such pair of ideals can be combined to give an integer

$$[\theta^{le\gamma} - f(\alpha)]\,\theta^{E(sle\gamma\mu)-1} \text{ which} \equiv 0 \pmod{\mathfrak{p}^{m+s+2}}.$$

The modulus (64) is therefore the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$).

113. XVII. $\mathfrak{p}$ odd, $\mathfrak{p}=\mathfrak{p}_j$ of (34), $g>e\geqslant p$; $p\equiv 0 \pmod{\mathfrak{p}^g}$ exactly, $e\epsilon = 1$.

The diagram of squares shews that

$$^{q}C_{g}\,(\alpha-b)^{g}b^{q-g}+(b^{q}-b)\equiv 0 \pmod{\mathfrak{p}^{jg+2}}$$

whence, since $\qquad {}^{q}C_{g}\equiv p^{j-1} \pmod{p^{j}},$

$$(\alpha-b)^{g}\equiv(1-b^{p-1})/p^{j-1}\equiv Bp \pmod{\mathfrak{p}^{g+2}},$$

B being a rational integer prime to p. Noting that

$$\theta^{l},\quad (\alpha-b)^{mg\epsilon},\quad \mathfrak{p}^{mg\epsilon}$$

are isobaric (mod $\mathfrak{p}$), we put

$$\omega=\phi(\theta)=\theta^{l}-A\,(\alpha-b)^{mg\epsilon},$$

giving $\quad \omega^{e}+\sum_{1,\,e-1}{}^{e}C_{i}A^{i}(\alpha-b)^{mg\epsilon i}\omega^{e-i}+A^{e}(\alpha-b)^{mg}-p^{m}\alpha=0.$

Since $\qquad A^{e}(\alpha-b)^{mg}-p^{m}\alpha\equiv p^{m}(AB^{m}-\alpha) \pmod{\mathfrak{p}^{mg+2}},$

the number ω is isobaric with $\mathfrak{p}^{(mg+1)\epsilon}$ (mod $\mathfrak{p}$) when

$$A=bB^{p-1-m},$$

but with no higher power of $\mathfrak{p}$ for any value of A in $[\alpha]$. It follows as in previous cases that the $(1+s)$th term of the stem of $\boldsymbol{\Theta}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o}\theta^{x_s}\,[\theta^{l}-bB^{p-1-m}(\alpha-b)^{mg\epsilon}]^{y_s}/\mathfrak{p}^{s}$$

where x_s, y_s are so chosen that x_s+ly_s is least, not exceeding $le-1$, subject to the inequality

$$(mg\epsilon/l)\,x_s+(mg+1)\,\epsilon y_s\geqslant s.$$

114. XVIII. p odd, $\mathfrak{p}=\mathfrak{p}_j$ of (34), $e\geqslant g\geqslant p$;

$$e=g^{f}r',\quad 1\leqslant r'<g=rr',\quad e\epsilon=g\gamma=1.$$

It is easily proved, as in XVII, that

$$\omega_1\equiv\theta^{le\gamma}-bB^{p-1-m}(\alpha-b)^{m}\equiv\theta^{le\gamma}-A_0\,(\alpha-b)^{m}$$

is isobaric with $\mathfrak{p}^{m+\gamma}$ (mod $\mathfrak{p}$). Next putting

$$gu_1+mv_1=mg+1,\quad u_1\geqslant 1,\quad v_1\geqslant 1,$$

and $\qquad \omega_2=\theta^{le\gamma}-bB^{p-1-m}(\alpha-b)^{m}-A_1\,(\alpha-b)^{u_1}\theta^{le\gamma\gamma v_1},$

$$\omega_2^{g}\equiv p^{m}(\alpha-b)\,[1-A_1A_0^{v_1}B^{m}] \pmod{\mathfrak{p}^{mg+1+\gamma}} \text{ exactly},$$

so, if $\qquad A_1=A_0^{-v_1}B^{-m}\equiv b^{p-1-v_1}B^{mv_1-m} \pmod{p},$

ω_2 is isobaric with $\mathfrak{p}^{m+\gamma+\gamma\gamma}$ (mod $\mathfrak{p}$). Repeating the argument it is proved without difficulty that if

$$g^{i}u_i+m\,[v_1+gv_2+\ldots+g^{i-1}v_i]=g^{i}u_i+mw_i=mg^{i}+g^{i-1}+\ldots+1$$

and $\qquad A_i=(-)^{i-1}v_1^{\,i-1}b^{p-i}B^{-m}\{B^{m}b^{p-2}\}^{v_1+\ldots+v_i},\quad i\geqslant 1,$

then $\qquad \phi(\theta)\equiv\theta^{le\gamma}-A_0\,(\alpha-b)^{m}-\sum_{1,\,f-1}A_i\,(\alpha-b)^{u_i}\theta^{le\gamma^{i+1}w_i}$

is isobaric with $\mathfrak{p}^{m+\gamma+\gamma\gamma+\dots+r'\epsilon}$ (mod $\mathfrak{p}$). Further, when $r>1$ the number

$$\phi_1(\theta)\equiv\theta^{le\gamma r}-A_0(a-b)^{mr}-\sum_{1,f-1}A_i(a-b)^{ru_i}\theta^{ler\gamma^{i+1}w_i}-A_f(a-b)^{u_f}\theta^{lw_f}$$

is isobaric with $\mathfrak{p}^{rm+r\gamma+r\gamma\gamma+\dots+\epsilon}$ (mod $\mathfrak{p}$).

The $(1+s)$th term of the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o}\theta^x\phi(\theta)^y\phi_1(\theta)^z/\mathfrak{p}^s,$$

where, subject to the inequality

$$xmg\epsilon/l+y(m+\gamma+\dots+\gamma^f)+zr(m+\gamma+\dots+\gamma^{f+1})\geqslant s,$$

$x+yle+zler$ is least—not exceeding $le-1$.

115. XIX. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $g>e\geqslant 2$; and

XX. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $e\geqslant g>2$.

The prime ideal $\mathfrak{p}$ is of the second degree and attached to the last edge E_{j-1} of the diagram, passing through marked nodes which correspond to the terms

$$\{(a-b)^{2g}+2(a-b)^g+2^2\}\,2^{j-2}.$$

Since there is no marked node on the integer line nearest to E_{j-1}, it follows that

$$(a-b)^{2g}+2(a-b)^g+2^2\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^{2g+2})\quad\dots\dots\dots(65),$$

$$(a-b)^{3g}\equiv 2^3\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^{3g+2}),$$

and

$$(a-b)^{3gt}-2^{3t}\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^{3gt+2}),\quad t>1.$$

Again when $m\equiv 1$ (mod 3)

$$(a-b)^{mg}\equiv 2^{m-1}(a-b)^g\ (\mathrm{mod}\ \mathfrak{p}^{mg+2})$$

and

$$(a-b)^{2mg}\equiv 2^{m-2}(a-b)^{2g}\ (\mathrm{mod}\ \mathfrak{p}^{2mg+2})$$

whence

$$\left.\begin{array}{r}(a-b)^{2mg}+2^m(a-b)^{mg}+2^{2m}\\ \equiv 2^{2m-2}[(a-b)^{2g}+2(a-b)^g+2^2]\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^{2mg+2})\end{array}\right\}\ \dots(66).$$

The congruence (66) also holds when $m\equiv 2$ (mod 3).

Next putting

$$\Omega=\Omega_{j-1}=\frac{b-b^q}{2^{j-1}(a^g-b^g)}\quad\dots\dots\dots\dots\dots\dots\dots\dots(67)$$

and noting, from (39), that $\Omega\not\equiv 0$ (mod $\mathfrak{p}$), the numbers

$$0,\ 1,\ \Omega,\ 1+\Omega$$

form a complete set of residues (mod $\mathfrak{p}$) and satisfy

$$\Omega^2+\Omega+1\equiv 0\ (\mathrm{mod}\ \mathfrak{p}).$$

Also

$$\left.\begin{array}{l}(a-b)^g\equiv(a^2-b^2)^{g/2}\equiv\\ \quad\dots\equiv(a^{g/2}-b^{g/2})^2\equiv(a^{g/2}-b^{g/2})(a^{g/2}+b^{g/2})\equiv a^g-b^g\ (\mathrm{mod}\ \mathfrak{p}^{g+2})\end{array}\right\}\ (68),$$

hence, from (67) and (65) above,

$$\left.\begin{aligned} &(\alpha-b)^g\,\Omega\equiv 2 \pmod{\mathfrak{p}^{g+2}} \\ \text{and}\qquad &\Omega^2+\Omega+1\equiv 0 \pmod{\mathfrak{p}^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(69).$$

116. XIX. $g > e \geqslant 2$. θ^l is isobaric with $(\alpha-b)^{mg\epsilon}$ (mod $\mathfrak{p}$) and if

$$\phi(\theta)=\omega=\theta^l-\Omega^z(\alpha-b)^{mg\epsilon},$$

then $$\omega^e+\sum_{1,\,e-1}{}^{e}C_i\,\Omega^{zi}(\alpha-b)^{mg\epsilon i}\,\omega^{e-i}+\Omega^{ez}(\alpha-b)^{mg}-2^m\alpha=0.$$

When $z=me$,

$$N_a(\omega)\equiv\Omega^m(\alpha-b)^{mg}-2^m\alpha\equiv\Omega^m(\alpha^g-b^g)^m-2^m\alpha\equiv 2^m(1-\alpha) \pmod{\mathfrak{p}^{mg+2}}$$

from (68) and (69) above. Hence

$$\phi(\theta)\equiv\theta^l-\Omega^{me}(\alpha-b)^{mg\epsilon}$$

is isobaric with $\mathfrak{p}^{mg\epsilon+\epsilon}$ (mod $\mathfrak{p}$), and, as in former cases, the $(1+s)$th term of the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o}\,\theta^x\phi(\theta)^y/\mathfrak{p}^s$$

where x, y are integers ($\geqslant 0$) making $x+ly$ least subject to the inequality

$$xmg\epsilon/l+y(mg\epsilon+\epsilon)\geqslant s.$$

117. XX. $e\geqslant g>2$; $e=g^fr'$, $1\leqslant r'<g=rr'$, $g\gamma=e\epsilon=1$.

The number $\theta^{le\gamma}+\Omega^{mg}(\alpha-b)^m$ is isobaric with $\mathfrak{p}^{m+\gamma}$ (mod $\mathfrak{p}$). Choosing the unique set of rational integers u_i, v_i, w_i which satisfy

$$\begin{aligned} &mg^i+g^{i-1}+\ldots+g+1 \\ &=g^iu_i+m\left[v_1+gv_2+\ldots+g^{i-1}v_i\right] \\ &=g^iu_i+mw_i,\quad u_i>0,\ v_i\geqslant 0,\ w_i>0, \end{aligned}$$

it is easily proved that

$$\phi(\theta)\equiv\theta^{le\gamma}+\Omega^{mg}(\alpha-b)^m+\sum_{1,\,f-1}\Omega_i(\alpha-b)^{u_i}\theta^{le\gamma^{i+1}w_i}$$

is isobaric with $\mathfrak{p}^{m+\gamma+\gamma\gamma+\ldots+r'\epsilon}$ (mod $\mathfrak{p}$), provided $\Omega_i=1$, Ω, Ω^2 according as

$$m(g-w_i)\equiv 0,\ 1,\ 2 \pmod 3.$$

Further

$$\phi_1(\theta)\equiv\theta^{le\gamma r}+\Omega^{mgr}(\alpha-b)^{mr}+\sum_{1,\,f-1}\Omega_i^{\,r}(\alpha-b)^{ru_i}\theta^{ler\gamma^{i+1}w_i}+\Omega_f(\alpha-b)^{u_f}\theta^{lw_f}$$

is isobaric with $\mathfrak{p}^{mr+\gamma r+\ldots+\epsilon}$ (mod $\mathfrak{p}$).

Accordingly the $(1+s)$th term of the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o}\,\theta^x\phi(\theta)^y\phi_1(\theta)^z/\mathfrak{p}^s$$

where $x+le\gamma y+le\gamma rz$ is least subject to

$$xm/le\gamma+y\left[m+\gamma+\ldots+\gamma^f\right]+zr\left[m+\gamma+\ldots+\gamma^{f+1}\right]\geqslant s.$$

118. XXI. $p=2$, $\mathfrak{p}=\mathfrak{p}_{j-1}$ of (35), $e \geqslant g=2$; $g\gamma=m\mu=1$.

An inspection of the diagram of squares shews that

$$(a-b)^4+2(a-b)^2+2^2+2^2(a-b)\equiv 0 \pmod{\mathfrak{p}^6},$$

i.e.

$$(a^2-b^2)^2+2(a^2-b^2)+2^2\equiv 0 \pmod{\mathfrak{p}^6} \quad\ldots\ldots\ldots\ldots(70).$$

Putting

$$\Omega=\Omega_{j-1}=\frac{b-b^q}{2^{j-1}(a^2-b^2)},$$

$$\Omega(a^2-b^2)=(b-b^q)/2^{j-1}\equiv 2 \pmod{2^2} \text{ or } \pmod{\mathfrak{p}^4},$$

whence, from (70),

$$\Omega^2+\Omega+1\equiv 0 \pmod{\mathfrak{p}^2}.$$

Next if

$$\omega=\phi(\theta)=\theta^{le\gamma}-\Omega^{2m}(a-b)^m,$$

$$\omega^2+2\Omega^{2m}(a-b)^m\omega+\Omega^{4m}(a-b)^{2m}-2^m a=0,$$

and since

$$\begin{aligned}\Omega^{4m}(a-b)^{2m}-2^m a &\equiv \Omega^m(a-b)^m[(a+b)-2b]^m-2^m a\\ &\equiv \Omega^m(a-b)^m[(a+b)^m-2mb(a+b)^{m-1}]-2^m a\\ &\equiv 2^m(b-a)(1+mb\Omega) \pmod{\mathfrak{p}^{2m+2}},\end{aligned}$$

$\phi(\theta)$ is isobaric with $\mathfrak{p}^{m+\gamma}$ (mod $\mathfrak{p}$). Hence the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o},\ \frac{\mathfrak{o}\theta^{E(le\gamma\mu)}}{\mathfrak{p}},\ \frac{\mathfrak{o}\theta^{E(2le\gamma\mu)}}{\mathfrak{p}^2},\ \ldots\ \frac{\mathfrak{o}\theta^{E(le\gamma-le\gamma\mu)}}{\mathfrak{p}^{m-1}},\ \frac{\mathfrak{o}\phi(\theta)}{\mathfrak{p}^m},$$

$$\frac{\mathfrak{o}\theta^{E(le\gamma^2\mu)}\phi(\theta)}{\mathfrak{p}^{m+1}},\ \frac{\mathfrak{o}\theta^{E(le\gamma\mu+le\gamma^2\mu)}\phi(\theta)}{\mathfrak{p}^{m+2}},\ \ldots\ \frac{\mathfrak{o}\theta^{E(le\gamma-le\gamma^2\mu)}\phi(\theta)}{\mathfrak{p}^{2m}},$$

the last term (at least) being omitted when $4m>le$.

119. XXII. $\mathfrak{p}=\mathfrak{p}_h$ of (36); $\rho^{-1}=m(p-1)$.

Since $\mathfrak{p}$ divides p^m just $m(p-1)$ times the argument given in v shews that the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o},\ \frac{\mathfrak{o}\theta^{E(le\rho)}}{\mathfrak{p}},\ \frac{\mathfrak{o}\theta^{E(2le\rho)}}{\mathfrak{p}^2},\ \ldots\ \frac{\mathfrak{o}\theta^{E(le-le\rho)}}{\mathfrak{p}^{m-1}}.$$

120. XXIII. $\mathfrak{p}=\mathfrak{p}_{h+1}$ of (36); $m\mu=1$.

This prime ideal divides p once only: hence, as in v, the stem of $\mathfrak{O}$ (bas $\mathfrak{p}$) is

$$\mathfrak{o},\ \frac{\mathfrak{o}\theta^{E(le\mu)}}{\mathfrak{p}},\ \frac{\mathfrak{o}\theta^{E(2le\mu)}}{\mathfrak{p}^2},\ \ldots\ \frac{\mathfrak{o}\theta^{E(le-le\mu)}}{\mathfrak{p}^{m-1}}.$$

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OPERATIONAL METHODS IN MATHEMATICAL PHYSICS

BY

HAROLD JEFFREYS

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"Even Cambridge mathematicians deserve justice."

OLIVER HEAVISIDE

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PREFACE

It is now over thirty years since Heaviside's operational methods of solving the differential equations of physics were first published, but hitherto they have received very little attention from mathematical physicists in general. The chief reason for this lies, I think, in the lack of a connected account of the methods. Heaviside's own work is not systematically arranged, and in places its meaning is not very clear. Bromwich's discussion of his method by means of the theory of functions of a complex variable established its validity; and as a matter of practical convenience there can be little doubt that the operational method is far the best for dealing with the class of problems concerned. It is often said that it will solve no problem that cannot be solved otherwise. Whether this is true would be difficult to say; but it is certain that in a very large class of cases the operational method will give the answer in a page when ordinary methods take five pages, and also that it gives the correct answer when the ordinary methods, through human fallibility, are liable to give a wrong one. In particular, when we discuss the small oscillations of a dynamical system with n degrees of freedom by the method of normal coordinates, we obtain a determinantal equation of the nth degree to give the speeds of the normal modes. To find the ratios of the amplitudes we must then complete the solution for each mode. If we want the actual motion due to a given initial disturbance we must solve a further family of $2n$ simultaneous equations, unless special simplifying circumstances are present. In the operational method a formal operational solution is obtained with the same amount of trouble as is needed to give the period equation in the ordinary method, and from this the complete solution is obtainable at once by a general rule of interpretation. For continuous systems the advantage of the operational method is even greater, for it gives both periods and amplitudes easily in problems where the amplitudes cannot be found by the ordinary method without a knowledge of some theorem of expansion in normal functions analogous to Fourier's theorem. Heat conduction is also especially conveniently treated by operational methods.

Since Bromwich's discussion it has often been said that the operational method is only a shorthand way of writing contour integrals. It may be;

but at least one may reply that a shorthand that avoids the necessity of writing $\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} d\kappa$ in every line of the work is worth while. Connected with the saving of writing, and perhaps largely because of it, is the fact that the operational mode of attack seems much the more natural when one has any familiarity with it. After all, the use of contour integrals in this connexion was introduced by Bromwich, who has repeatedly declared that the direct operational method of solution is the better of the two.

My own reason for writing the present work is mainly that I have found Heaviside's methods useful in papers already published, and shall probably do so again soon, and think that an accessible account of them may be equally useful to others. In one respect I must offer an apology to the reader. Heaviside developed his methods mostly in relation to the theory of electromagnetic waves. Having myself no qualification to write about electromagnetic waves I have refrained from doing so; but as the operators occurring in the theory of these waves are mostly of types treated here I think the loss will not be serious. It can in any case be remedied by reading Heaviside's works or some of the papers in the list at the end of this tract.

A chapter on dispersion has been included. The operational solution can be translated instantly into a complex integral adapted for evaluation by the method of steepest descents; a short account of the latter method has also been given, because it is not at present very accessible, and is often incorrectly believed to be more difficult than the method of stationary phase. Two cases where the Kelvin first approximation to the wave form breaks down are also discussed.

My indebtedness to the writings of Dr Bromwich is evident from the references in the text. In addition, the problems of 4.4 and 4.5 are taken directly from his lecture notes, and several others are included largely as a result of conversations with him.

My thanks are also due to the staff of the Cambridge University Press for their care and consideration during publication.

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CHAPTER I

FUNDAMENTAL NOTIONS

1.1. Let us consider the linear differential equation of the first order

$$\frac{dy}{dx}=Ry+S, \quad \text{..........................(1)}$$

where R and S are known functions of x, bounded and integrable when $0 \leqslant x \leqslant a$. Suppose further that $y=y_0$ when $x=0$. Let Q denote the operation of integrating with regard to x from 0 to x, so that

$$Qy=\int_0^x y\,dx. \quad \text{.............................(2)}$$

Perform the operation Q on both sides of the equation (1). Then we find

$$y-y_0=Q(Ry+S), \quad \text{......................(3)}$$

and the right side vanishes with x. This can be rewritten in the forms

$$(1-QR)\,y=y_0+QS, \quad \text{......................(4)}$$

$$y=y_0+QS+QRy. \quad \text{..........................(5)}$$

These are both equivalent to the original differential equation (1) together with the given terminal condition. When we write QRy we mean, of course, that R is to be multiplied into y and the product integrated with regard to x from 0 to x. But the whole expression for y may be substituted in the last term of (5), giving in succession

$$\begin{aligned} y&=y_0+QS+QR(y_0+QS+QRy)\\ &=y_0+QS+QR(y_0+QS)+QRQR(y_0+QS+QRy)\\ &=(y_0+QS)+QR(y_0+QS)+QRQR(y_0+QS)\\ &\qquad +QRQRQR(y_0+QS)+\dots \quad \text{..................(6)}\end{aligned}$$

on repeating the substitution indefinitely. We have to show that the infinite series (6) converges and that it is the correct solution. In evaluating each term R is supposed to be multiplied into the whole expression after it, and Q to operate on the whole expression after it. Suppose that within the range considered

$$|R|\leqslant A,\ |y_0+QS|\leqslant B, \quad \text{....................(7)}$$

where A and B are finite. Then the absolute value of the second term is less than ABx, that of the third than $A^2Bx^2/2!$, that of the fourth than $A^3Bx^3/3!$, while the general term is less than $A^nBx^n/n!$. The series therefore converges at least as fast as the power series for exp Ax. It therefore represents a definite function, and on substituting it in (5) we see that the equation is satisfied for all values of x; also the solution reduces to y_0 when $x=0$, as it should. Thus (6) is the correct solution.

The solution can also be written

$$y=(1+QR+QRQR+QRQRQR+\dots)(y_0+QS). \quad \text{......(8)}$$

The operator between the first pair of brackets is the binomial expansion of $(1-QR)^{-1}$, carried out as if QR was merely a number. Since y_0+QS is a determinate function, we can write the solution in the form

$$y=\frac{1}{1-QR}(y_0+QS), \quad \text{........................(9)}$$

provided that y_0+QS is evaluated first and that the operator $(1-QR)^{-1}$ is expanded by the binomial theorem before interpretation. In fact (9) is merely a shorthand rule for writing (8). But on returning to (4) we see that (9) is also the solution of (4) carried out as if $1-QR$ was a mere number.

It is evident from (8) that the values of S for negative values of x do not affect the solution provided y_0 is kept the same. Suppose then that S was zero for all negative values of x, and that y was zero when $x=0$. The solution would be

$$y=(1+QR+QRQR+\dots)QS, \quad \text{..............(10)}$$

and this solution would be unaltered if the lower limits of all the integrals were replaced by $-\infty$. But if now we add to S a constant y_0/ξ for all values of x between 0 and ξ, QS will be increased by y_0 for all values of x greater than ξ, and (10) will be converted into (8). If then ξ tends to zero, y_0 remaining the same, y_ξ will tend to y_0, and we recover the solution (8) with the original initial condition. The physical interest of this result is that it corresponds to our notions of causality. Suppose that the independent variable x is the time, and that y represents the departure of some variable from its equilibrium value; then R represents a property of the system and S an external disturbing influence. If the system was originally in its equilibrium state, the form (10) exhibits the disturbance produced by the external influence after it enters. If it was undisturbed up to time zero, the part of (8) depending on S represents the effect of the finite disturbances acting at subsequent times, while the part depending on y_0 represents the effect of the impulsive disturbance at time 0 required to change y suddenly from zero to y_0. If we like we can separate the solution into two parts

$$(1-QR)^{-1}y_0 \text{ and } (1-QR)^{-1}QS,$$

and say that the first represents the effect of the initial conditions and the second that of the subsequent disturbances.

1.11. The method just given can be extended easily to cover many equations of the second and higher orders. Thus if our equation is

$$\frac{d^2y}{dx^2}=Ry+S, \quad \dots\dots(1)$$

with $y=y_0$ and $\frac{dy}{dx}=y_1$ when $x=0$, we find by integration

$$\frac{dy}{dx}-y_1=Q(Ry+S), \quad \dots\dots(2)$$

$$y-y_0-xy_1=QQ(Ry+S). \quad \dots\dots(3)$$

This leads to the solution in a series

$$y=(1+Q^2R+Q^2RQ^2R+\dots)(y_0+xy_1+Q^2S), \quad \dots\dots(4)$$

where $$Q^2f(x)=\int_0^x\int_0^\xi f(t)\,d\xi\,dt. \quad \dots\dots(5)$$

1.12. As a special example, consider the equation

$$\frac{dy}{dx}=ay, \quad \dots\dots(1)$$

with $y=1$ when $x=0$. Then carrying out the process of 1.1 (6) we get

$$y=(1+aQ+a^2Q^2+\dots)1 \quad \dots\dots(2)$$

$$=1+ax+\frac{a^2x^2}{2!}+\dots, \quad \dots\dots(3)$$

the ordinary expansion of exp ax. Or take the equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+x^2y=0, \quad \dots\dots(4)$$

with $y=1$ and $dy/dx=0$ when $x=0$. We can infer

$$y=1-Q\left\{\frac{1}{x}Q(xy)\right\} \quad \dots\dots(5)$$

$$=1-\left(\frac{x}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4-\left(\frac{1}{3!}\right)^2\left(\frac{x}{2}\right)^6+\dots, \quad \dots\dots(6)$$

the ordinary expansion of $J_0(x)$.

1.2. The foregoing method is due originally to J. Caqué*; it is a valuable practical method of obtaining numerical solutions of linear differential equations. Its extension to equations of any order, or to families of simultaneous equations of the first order, is more difficult unless special simplifications enter. But if the equations have constant coefficients, which is an extremely common case in physical applications, a considerable development is possible. This arises from the fact that the operator Q obeys the fundamental laws of algebra. Thus if a is a constant and u and v known functions of x,

$$Q(au) = a(Qu), \quad \text{.........(1)}$$

$$Q(u+v) = Qu + Qv, \quad \text{.........(2)}$$

$$Q^m Q^n u = Q^n Q^m u = Q^{m+n} u. \quad \text{.........(3)}$$

Consequently Q behaves in algebraic transformations just like a number. If for instance we have two operators $f(Q)$ and $g(Q)$ both expressible as sums of integral powers of Q, thus

$$f(Q) = a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 + \dots \quad \text{.........(4)}$$

$$g(Q) = b_0 + b_1 Q + b_2 Q^2 + b_3 Q^3 + \dots \quad \text{.........(5)}$$

where the a's and b's are constants, let us for a moment replace Q by a number z small enough to make the series converge absolutely. Form the product series

$$\begin{aligned} f(z)\,g(z) &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ &= c_0 + c_1 z + c_2 z^2 + \dots \quad \text{.........(6)} \end{aligned}$$

say. Consider first the case where the series are both polynomials. Then if S is an integrable function of x

$$f(Q)\,g(Q)\,S = (c_0 + c_1 Q + c_2 Q^2 + \dots)\,S. \quad \text{.........(7)}$$

For the left side means

$$\begin{aligned} &(a_0 + a_1 Q + a_2 Q^2 + \dots)(b_0 S + b_1 QS + b_2 Q^2 S + \dots) \\ &\quad = a_0(b_0 S + b_1 QS + b_2 Q^2 S + \dots) + a_1(b_0 QS + b_1 Q^2 S + \dots) \\ &\qquad + a_2(b_0 Q^2 S + \dots) + \dots \quad \text{.........(8)} \end{aligned}$$

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* *Liouville's Journal*, (2), 9 (1864), 185–222. Further developments are given by Fuchs, *Ann. d. Matem.*, (2), 4 (1870), 36–49; Peano, *Math. Ann.*, 32 (1888), 450–456; H. F. Baker, *Proc. Lond. Math. Soc.*, (1), 34 (1902), 347–360; (1), 35 (1902), 333–378; (2), 2 (1905), 293–296; *Phil. Trans.*, A, 216 (1916), 129–186. Caqué considers only a single differential equation, but notices that the operator in the result is in the form of a binomial expansion. For other operational methods based on these principles, but applicable to equations of order higher than the first, or to families of equations of the first order, the above papers of Prof. Baker may be consulted. Physical applications are given by W. L. Cowley and H. Levy, *Phil. Mag.*, 41 (1921), 584–607; Jeffreys, *Proc. Lond. Math. Soc.*, (2), 23 (1924), 454 and 465; *M.N.R.A.S.*, Geoph. Suppl. 1 (1926), 380–383.

using (1) and (2); and then using (3) we can collect the terms involving the same power of Q and obtain

$$a_0 b_0 S + (a_0 b_1 + a_1 b_0) QS + (a_0 b_2 + a_1 b_1 + a_2 b_0) Q^2 S + \ldots \quad (9)$$

which is by definition the same as

$$(c_0 + c_1 Q + c_2 Q^2 + \ldots) S. \quad \ldots\ldots\ldots\ldots(10)$$

When the series are infinite, their convergence may be established easily. If the series for $f(z)$ converges absolutely when $|z| \leqslant r$, then a number M must exist such that $a_n r^n \leqslant M$ for all positive integral values of n. Thus

$$a_n \leqslant \frac{M}{r^n}, \quad \ldots\ldots\ldots\ldots(11)$$

and also if for all values of x, $|S| \leqslant C$, where C is a constant,

$$Q^n S \leqslant C \frac{x^n}{n!}. \quad \ldots\ldots\ldots\ldots(12)$$

Hence if

$$f(Q) S = \sum_{n=0}^{\infty} a_n Q^n S, \quad \ldots\ldots\ldots\ldots(13)$$

each term is less than the corresponding term of the series

$$\sum_{n=0}^{\infty} M \left(\frac{x}{r}\right)^n \frac{C}{n!}, \quad \ldots\ldots\ldots\ldots(14)$$

which converges for all values of x however large. So long as $f(z)$ is expansible about the origin in a convergent power series, however small its radius of convergence may be, the expression $f(Q) S$ will be the sum of an absolutely convergent series however great x may be. If $f(z)$ and $g(z)$ are both expansible within some circle, their product series will also converge within this circle and the expression $f(Q) g(Q) S$ will be an absolutely convergent series however great x may be. Provided with this result we can now easily extend the result (7) to the case where $f(z)$ and $g(z)$ are infinite series, by methods analogous to those used to justify the multiplication of two absolutely convergent power series.

1.3. We can now extend these methods to the solution of a family of n simultaneous differential equations of the first order with constant coefficients. Suppose the equations are

$$\left.\begin{array}{l} e_{11} y_1 + e_{12} y_2 + \ldots + e_{1n} y_n = S_1 \\ e_{21} y_1 + e_{22} y_2 + \ldots + e_{2n} y_n = S_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_{n1} y_1 + e_{n2} y_2 + \ldots + e_{nn} y_n = S_n \end{array}\right\}, \quad \ldots\ldots\ldots\ldots(1)$$

where the y's are dependent variables, x the independent variable, e_{rs} denotes $a_{rs}\dfrac{d}{dx}+b_{rs}$, where a_{rs} and b_{rs} are constants, and the S's are known functions of x. We do not assume that

$$a_{rs}=a_{sr},\quad b_{rs}=b_{sr},$$

but we do assume that the determinant formed by the a's is not zero. When $x=0$, $y_1=u_1$, and so on, where the u's are known constants.

First perform the operation Q on both sides of each equation. We have

$$\begin{aligned} Qe_{rs}y_s &= \int_0^x\left(a_{rs}\frac{dy_s}{dx}+b_{rs}y_s\right)dx \\ &= a_{rs}(y_s-u_s)+b_{rs}Qy_s \\ &= f_{rs}y_s-a_{rs}u_s, \end{aligned} \qquad \text{.........................(2)}$$

where f_{rs} denotes the operator $a_{rs}+b_{rs}Q$. Then the equations and the initial conditions are together equivalent to the equations

$$\left.\begin{array}{l} f_{11}y_1+f_{12}y_2+\dots+f_{1n}y_n=v_1+QS_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{n1}y_1+f_{n2}y_2+\dots+f_{nn}y_n=v_n+QS_n \end{array}\right\}, \qquad \text{............(3)}$$

where
$$v_r=a_{r1}u_1+a_{r2}u_2+\dots+a_{rn}u_n. \qquad \text{..................(4)}$$

The general equation can be written compactly

$$\Sigma_s f_{rs}y_s=v_r+QS_r. \qquad \text{........................(5)}$$

Now let D denote the operational determinant formed by the f's, namely,

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}. \qquad \text{.....................(6)}$$

If this determinant is expanded by the ordinary rules of algebra and equal powers of Q collected, we shall obtain a polynomial in Q. The term independent of Q is simply the determinant formed by the a's, which by hypothesis does not vanish. Now let F_{rs} denote the minor of f_{rs} in this determinant, taken with its proper sign. F_{rs} also is a polynomial in Q.

Now operate on the first of (3) with F_{1s}, on the second with F_{2s}, and so on, and add. Then in the sum the operator acting on y_m, say, is

$$\Sigma_r F_{rs}f_{rm}. \qquad \text{..............................(7)}$$

If $m=s$, this sum is the determinant D; if $m\neq s$, it is a determinant with two columns equal and therefore is zero. The resulting equation is therefore

$$Dy_s=\Sigma_r F_{rs}(v_r+QS_r). \qquad \text{.....................(8)}$$

Now if all the S's are bounded and integrable within the range of values of x contemplated, the expression on the right of (8) is also a bounded integrable function of x. Also since the function $D(z)$, obtained by replacing Q in D by a number z, is regular and not zero at $z=0$, the function $1/D(z)$ is expressible as a power series in z with a finite radius of convergence. Define D^{-1} as the power series in Q obtained by putting Q for z in the series for $1/D(z)$. Then operate on both sides of (8) with D^{-1}. We have

$$D^{-1}Dy_s = D^{-1}\Sigma_r F_{rs}(v_r + QS_r). \quad\text{..................(9)}$$

But, since series of positive integral powers of Q can be multiplied according to the rules of algebra, $D^{-1}D$ gives simply unity, and we have the solution

$$y_s = D^{-1}\Sigma_r F_{rs}(v_r + QS_r). \quad\text{.....................(10)}$$

This gives a complete formal solution of the problem. Its form is often convenient for actual computation, especially for small values of x; but it can also be expressed in finite terms. D is, as we have seen, a polynomial in Q of degree n at most, while F_{rs} is a polynomial in Q of degree $n-1$ at most. Our solution is therefore of the form

$$y_s = \frac{\phi(Q)}{D(Q)} + \frac{\psi(Q)}{D(Q)} QS_r, \quad\text{..................(11)}$$

where ϕ and ψ are polynomials whose degree is ordinarily one less than that of D. Since the determinant formed by the a's is not zero we may denote it by A, and then D will be the product of n linear factors, thus

$$D = A(1-\alpha_1 Q)(1-\alpha_2 Q)\dots(1-\alpha_n Q), \quad\text{...........(12)}$$

where the α's will ordinarily be all different. Then $\phi(Q)/D(Q)$ can be expressed as the sum of a number of partial fractions of the form

$$\frac{L}{1-\alpha Q}. \quad\text{..............................(13)}$$

But by 1.12 (3) this is the same as $Le^{\alpha x}$. The part of the solution arising from the v's can therefore be expressed as a linear combination of exponentials.

The justification of the decomposition of $\phi(Q)/D(Q)$ into partial fractions is that this decomposition is a purely algebraic process. Hence if the partial fractions and the original operator are all expanded in positive powers of Q, and like powers of Q are collected, the coefficients of a given power of Q will be the same in both expressions, and the equivalence is complete.

Exceptional cases will occur if D contains no term in Q^n, or if two or more of the α's are equal. If the term in Q^n is absent the expansion of the operator in partial fractions will usually contain a constant term. If the term in Q^{n-1} is also absent we must divide out, and the expansion in partial fractions will contain a term in Q; this will give a term in x on interpretation.

If several of the α's are equal, the expression in partial fractions will involve terms of the form $M(1-\alpha Q)^{-r}$. These can be interpreted by direct expansion; but another method is more convenient. Starting with

$$(1-\alpha Q)^{-1}=e^{\alpha x}, \quad \text{.........................(14)}$$

let us differentiate $r-1$ times with regard to α. We find

$$\frac{(r-1)!\,Q^{r-1}}{(1-\alpha Q)^r}=x^{r-1}e^{\alpha x}, \quad \text{....................(15)}$$

so that

$$\frac{Q^{r-1}}{(1-\alpha Q)^r}=\frac{x^{r-1}}{(r-1)!}e^{\alpha x}. \quad \text{....................(16)}$$

A rather different form of resolution into partial fractions from the ordinary one is therefore necessary if each fraction is to give a single term in the solution. Instead of having constants in all the numerators we must have powers of Q, the power of Q needed in any fraction being one less than the degree of the denominator. But if we write p^{-1} for Q the fraction in (16) is algebraically equivalent to $p/(p-\alpha)^r$. In this form the power of p in the numerator is independent of the degree of the denominator, and it is easier to resolve into partial fractions of this form than into those involving Q directly.

The above remarks apply to the interpretation of the effect of the initial conditions; that is, of the first term on the right of (11). To find the effect of the terms in the S's we may expand the operators similarly. To interpret $(1-\alpha Q)^{-1}QS$ in finite terms, we note that it is the solution of $\frac{dy}{dx}-\alpha y=S$ that vanishes with x. This is easily found by the ordinary method to be

$$y=e^{\alpha x}Q(Se^{-\alpha x}), \quad \text{.........................(17)}$$

which is the interpretation required.

To sum up, we can solve a family of equations of the type (1) by first integrating each equation once from 0 to x with regard to x, allowing for the initial conditions. This gives a set of equations of the type (3). The subsequent process for deducing the operational solution (10) is exactly the same as if the operators f_{rs} were numbers, and the ordinary rules of algebra were applied. The solution can be evaluated by expanding the operators in ascending powers of Q and evaluating term by term; this in general gives an infinite series. Alternatively it can be obtained by resolving into partial fractions and interpreting each fraction separately; this gives an explicit solution in finite terms.

1.4. Heaviside's method is equivalent to that just given; it differs in using another notation, which is not quite so convenient in a formal proof of the theorem, but is rather more convenient for actual application. In Heaviside's notation the operator above called Q is denoted by p^{-1}. At present we need not specify the meaning of positive powers of p; negative integral powers are defined by induction, so that p^{-n} denotes Q^n. We have noticed that the passage from 1.3 (3) to 1.3 (10) is a purely algebraic process. Consequently if all the equations (3) were multiplied by constants before the algebraic solution the same answer would be obtained. Suppose then that we write the general equation 1.3 (5) in the form

$$\Sigma_s (a_{rs} + b_{rs} p^{-1}) y_s = \Sigma_s a_{rs} u_s + p^{-1} S_r. \qquad \text{..............(1)}$$

Multiply throughout by p as if this were a constant. We get

$$\Sigma_s (a_{rs} p + b_{rs}) y_s = \Sigma_s a_{rs} p u_s + S_r. \qquad \text{..............(2)}$$

On solving the n equations of this form we shall obtain a solution identical with 1.3 (10) except that p^{-1} will appear for Q, and both numerator and denominator will be multiplied by the same power of p. If the operators in the solution are expanded in negative powers of p, and p^{-1} is then interpreted as Q, the result will be identical with that already given. Comparing (2) with the original equations 1.3 (1) we see that the new form of our rule is as follows:

Write p for d/dx on the left of each equation; to the right of each equation add the result of dropping the b's on the left and replacing the y's by their initial values; solve the resulting equations (2) by algebra as if p was a number; and evaluate the result by expanding in negative powers of p and interpreting p^{-1} as the operation of integrating from 0 to x.

This is Heaviside's rule. In what follows the equations (2) will usually be called the subsidiary equations.

To obtain the solution explicitly, we put e_{rs} for $a_{rs}p + b_{rs}$, denote our determinant

$$\begin{vmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{vmatrix}$$

by Δ, and denote the minor of e_{rs} in this determinant, taken with its proper sign, by E_{rs}. Then the solution is

$$y_s = \Sigma_r \frac{E_{rs}}{\Delta}(p\Sigma_m a_{rm} u_m + S_r). \quad \text{.................(3)}$$

Since the determinant formed by the a's is not zero, Δ is of degree n in p, while E_{rs} is at most of degree $n-1$. The operators can therefore be expanded in negative powers of p, as we should expect; positive powers do not occur. All terms after the first vanish with x; the first is

$$\Sigma_r \frac{A_{rs}}{A} \Sigma_m a_{rm} u_m, \quad \text{.........................(4)}$$

where A_{rs} is the minor of a_{rs} in A. But

$$\left.\begin{aligned} \Sigma_r A_{rs} a_{rm} &= A \ \ (m = s) \\ &= 0 \ \ (m \neq s) \end{aligned}\right\}, \quad \text{.....................(5)}$$

and (4) reduces to u_s, as we should expect. This verifies that the solution satisfies the initial conditions.

1.5. To interpret the operational solution 1.4 (3) in finite terms we require rules for interpreting rational functions of p operating on unity and on other functions. We have already had the rules

$$p^{-1} = Q; \ p^{-2} = Q^2; \text{ and so on,} \quad \text{..................(1)}$$

$$p^{-1}1 = Q1 = x; \ p^{-2}1 = Qx = \frac{x^2}{2\,!}; \text{ and in general } p^{-n}1 = \frac{x^n}{n\,!}. \ \text{....(2)}$$

If unity is replaced by Heaviside's 'unit function,' here denoted by $H(x)$, which is zero for all negative values of x and 1 for all positive values, we shall still have

$$p^{-n} H(x) = \frac{x^n}{n\,!}, \quad \text{.........................(3)}$$

when x is positive, but it will vanish when x is negative. We can also replace the lower limit of the integrations by $-\infty$ without altering this interpretation.

Again, we shall have when x is positive

$$\frac{p}{p-a} = \frac{1}{1-aQ} = e^{ax}, \quad \text{........................(4)}$$

$$\frac{p}{(p-a)^n} = \frac{Q^{n-1}}{(1-aQ)^n} = \frac{x^{n-1}}{(n-1)\,!} e^{ax}, \quad \text{...............(5)}$$

$$\frac{a}{p-a} = \frac{p}{p-a} - 1 = e^{ax} - 1, \quad \text{....................(6)}$$

where the function operated on may be either unity or $H(x)$. In the latter case all the operators will give zero for negative values of x.

The operators in 1.4 (3) are of the form $f(p)/F(p)$, where $f(p)$ and $F(p)$ are polynomials in p, and $f(p)$ is of the same or lower degree than $F(p)$. If $F(p)$ is of degree n it can be resolved into n linear factors of the form $p-\alpha$. Then provided that the α's are all different and none of them zero we have the algebraic identity

$$\frac{f(p)}{pF(p)}=\frac{f(0)}{pF(0)}+\Sigma_\alpha \frac{f(\alpha)}{\alpha F'(\alpha)}\frac{1}{p-\alpha}, \quad\ldots\ldots\ldots\ldots\ldots(7)$$

whence
$$\frac{f(p)}{F(p)}=\frac{f(0)}{F(0)}+\Sigma_\alpha \frac{f(\alpha)}{\alpha F'(\alpha)}\frac{p}{p-\alpha}. \quad\ldots\ldots\ldots\ldots\ldots(8)$$

If this operates on unity or $H(x)$ we have therefore for positive values of x

$$\frac{f(p)}{F(p)}1=\frac{f(0)}{F(0)}+\Sigma_\alpha \frac{f(\alpha)}{\alpha F'(\alpha)}e^{\alpha x}. \quad\ldots\ldots\ldots\ldots\ldots(9)$$

To justify this we notice as before that (8) is a purely algebraic identity, and therefore if both sides are expanded in negative powers of p, beginning with constant terms, the expansions of the two sides will be identical, and on interpretation in terms of integrations will give the same result.

The formula (9) is usually known as Heaviside's expansion theorem; but as Heaviside's methods involve two other expansion theorems* it will be called the 'partial-fraction rule' in the present work.

If some of the α's are equal or zero, the expression (7) considered as a function of p will have a multiple pole, and its expression in partial fractions will contain terms of the form $(p-\alpha)^{-s}$, where s is an integer greater than unity, and α may be zero. Then $f(p)/F(p)$ will contain terms of the forms $p^{-(s-1)}$ or $p/(p-\alpha)^s$, which can be interpreted by means of (2) or (5).

By means of these rules we can evaluate all the expressions for the part of y_s in 1.4 (3) that depends on the initial values of the y's. If the S's are constants for positive values of x, as they often are, the same rules will apply to the part of the solution depending on them. If they are exponential functions such as $e^{\mu x}$, the easiest plan is usually to rewrite this as $p/(p-\mu)$ and reinterpret. Thus

$$\frac{f(p)}{F(p)}e^{\mu x}=\frac{f(p)}{F(p)}\frac{p}{p-\mu}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$=\frac{f(\mu)}{F(\mu)}e^{\mu x}+\Sigma\frac{f(\alpha)}{(\alpha-\mu)F'(\alpha)}e^{\alpha x}. \quad\ldots\ldots\ldots(11)$$

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* Namely, expansion in powers of Q or p^{-1} and interpretation term by term; and expansion in powers of e^{-ph}, where h is a constant, as in 4.2.

If S is expressed as a linear combination of exponentials we can apply this rule to each separately. This is applicable to practically all functions known to physics.

Alternatively we can resolve the operator acting on S into partial fractions and interpret $p^{-n}S$ by integration and other fractions by the rule of 1.3 (17)

$$\frac{1}{p-\alpha}S = e^{\alpha x}\int_0^x Se^{-\alpha x}\,dx. \qquad\qquad (12)$$

This completes our rules for solving a set of linear equations of the first order with constant coefficients. In comparison with the ordinary method, we notice that the rules are direct and lead immediately to a solution involving operators, which can then be evaluated completely by known rules. If it happens that we only require the variation of one unknown explicitly, we need not interpret the solutions for the others. In the ordinary method we have to find a complementary function and a particular integral separately. To find the former we assume a solution of the form $y_s = \lambda_s e^{\alpha x}$ and on substituting in the differential equations with the S's omitted we find an equation of consistency to determine the n possible values of α, and the ratios of the λ's corresponding to each. The particular integral is then found by some method, but it does not as a rule vanish with x. The actual values of the λ's are still undetermined, and the value associated with each α must be found by substituting in the initial conditions and again solving a set of n simultaneous equations. The labour of finding the equation for the α's and the ratios of the λ's corresponding to one of them is about the same as that of finding the operational solution in Heaviside's method; the rest of the work is avoided by the operational method. Further, if some of the α's are equal or zero considerable complications are introduced into the ordinary method, but not into the operational one.

1.6. The above work is applicable to all cases where A, the determinant formed by the a's, is not zero. We can show that if A is zero there is some defect in the specification of the system. A system is adequately specified if when we know the values of the dependent variables and the external disturbances the rates of change of the dependent variables are all determinate, and if the initial values of the variables are independent of one another. Now if A is zero, let us multiply the r'th equation by A_{rs}, the minor of a_{rs} in A, and add up for all values of r. The coefficient of dy_s/dx in the sum is A, which is zero; that of any other derivative is a determinant with two columns equal, and is also zero. Thus the derivatives disappear entirely from the sum, and we are left with a relation between the y's and the S's. If the coefficients of the y's also vanish, there must be a permanent relation between the S's, and one of the equations we started with is a mere logical consequence of the others; then we have not enough equations to determine the derivatives. If the coefficients of the y's do not vanish, we can put $x=0$ and obtain a relation between the initial values, which are therefore not independent.

1.7. The method is most easily extended to equations of higher order by breaking them up into equations of the first order. Thus if we have an equation of the second order such as

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=S, \tag{1}$$

we introduce a new variable z given by

$$\frac{dy}{dx}-z=0, \tag{2}$$

and the original equation can be replaced by

$$\frac{dz}{dx}+az+by=S. \tag{3}$$

We have now two equations of the first order in y and z. If initially $y=y_0$ and $dy/dx=y_1$, the subsidiary equations are

$$py-z=py_0, \tag{4}$$

$$(p+a)z+by=py_1+S. \tag{5}$$

Solving by algebra we find

$$y=\frac{(p^2+ap)y_0+py_1+S}{p^2+ap+b}. \tag{6}$$

To interpret, put

$$p^2+ap+b=(p-\alpha)(p-\beta), \tag{7}$$

and apply the partial-fraction rule. We find

$$y=\frac{1}{\alpha-\beta}(y_1-\beta y_0)e^{\alpha x}-\frac{1}{\alpha-\beta}(y_1-\alpha y_0)e^{\beta x}$$
$$+\frac{1}{\alpha-\beta}\left[e^{\beta x}\int_0^x Se^{-\beta x}\,dx-e^{\alpha x}\int_0^x Se^{-\alpha x}\,dx\right], \tag{8}$$

which is easily shown to satisfy all the conditions and therefore to be the solution required.

1.71. A few illustrative examples may be given. (p is written for d/dx from the start.)

1.
$$(p^2+4p+3)\,y=1;\ y_0=3;\ y_1=-2.$$

We consider the subsidiary equation

$$\begin{aligned}(p^2+4p+3)\,y&=1-2p+3\,(p^2+4p)\\&=3p^2+10p+1,\\ y&=\frac{3p^2+10p+1}{(p+1)(p+3)}\\&=\tfrac{1}{3}+3e^{-x}-\tfrac{1}{3}e^{-3x},\end{aligned}$$

on interpreting by the partial-fraction rule.

2.
$$(p^2+5p+6)\,y=12;\ y_0=2;\ y_1=0.$$

Consider
$$(p^2+5p+6)\,y=12+2\,(p^2+5p).$$

Cancelling the common factor,

$$y=2.$$

3.
$$(p+3)\,y=e^{-2x};\ y_0=0.$$

This can be written
$$(p+3)\,y=\frac{p}{p+2};$$

or
$$\begin{aligned}y&=\frac{p}{(p+2)(p+3)}\\&=e^{-2x}-e^{-3x}.\end{aligned}$$

4.
$$(p^2+n^2)\,y=0.$$

$$y=\frac{p^2\,y_0+py_1}{p^2+n^2}=y_0\cos nx+y_1\frac{\sin nx}{n}.$$

5.
$$(p+2)^2\,y=x^3\,e^{-2x};\ y_0=0;\ y_1=0.$$

The expression on the right is equivalent to the operator $2p/(p+2)^3$. Hence

$$y=\frac{2p}{(p+2)^5}=\frac{2x^4}{4\,!}\,e^{-2x}=\frac{1}{12}x^4\,e^{-2x}.$$

We notice the advantage of Heaviside's method in avoiding the use of simultaneous equations to determine the so-called 'arbitrary' constants of the ordinary method. In particular in the second example the data have a property leading to a simple solution. Heaviside's method seizes upon this immediately and gives the solution in one line; with the ordinary method the simultaneous equations for the constants would have to be solved as usual.

In general we may say that the more specific the problem the greater will be the convenience of the operational method.

1.72. The method just given can be extended easily to a family of equations of the second order. If the typical equation is again written

$$\Sigma_s e_{rs} y_s = S_r, \quad \dots\dots\dots\dots(1)$$

where now

$$e_{rs} = a_{rs}\frac{d^2}{dx^2} + b_{rs}\frac{d}{dx} + c_{rs}, \quad \dots\dots\dots\dots(2)$$

we can introduce n new variables $z_1, z_2, \dots z_n$ given by

$$\frac{dy_r}{dx} - z_r = 0, \quad \dots\dots\dots\dots(3)$$

thus treating the first derivatives of the y's as a set of new variables. Then the equations (1) are equivalent to

$$\Sigma_s a_{rs}\frac{dz_s}{dx} + \Sigma_s (b_{rs} z_s + c_{rs} y_s) = S_r, \quad \dots\dots\dots\dots(4)$$

and (3) and (4) constitute a set of $2n$ equations of the first order. Suppose also that when $x = 0$, $y_s = u_s$, $z_s = v_s$. According to our rule of 1.5 we must replace d/dx by p and add pu_r to the right of (3), and $\Sigma_s a_{rs} pv_s$ to the right of (4). We have now to solve by algebra, and may begin by writing the revised form of (3)

$$z_r = p(y_r - u_r). \quad \dots\dots\dots\dots(5)$$

When we substitute in the modified form of (4) we get

$$\Sigma_s a_{rs} p^2 (y_s - u_s) + \Sigma_s b_{rs} p (y_s - u_s) + \Sigma_s c_{rs} y_s = S_r + \Sigma_s a_{rs} pv_s, \quad (6)$$

or, on rearranging,

$$\Sigma_s (a_{rs} p^2 + b_{rs} p + c_{rs}) y_s = \Sigma_s (a_{rs} p^2 + b_{rs} p) u_s + \Sigma_s a_{rs} pv_s + S_r. \dots(7)$$

We have thus n equations for the y's to solve by algebra, and the solution can be interpreted by the rules.

In Bromwich's paper 'Normal Coordinates in Dynamical Systems' a method equivalent to the operational one is applied directly to a set of second order equations. First order equations, however, arise in some problems of physical interest, and merit a direct discussion. This, as we have seen, is easily generalized to equations of the second order.

1.73. In the discussion of the oscillations of stable dynamical systems by the ordinary method of normal coordinates there is a difficulty when the determinantal equation for the periods has equal roots. This does not arise in the present method. If the system is not dissipative, and the determinant formed by the e's as defined in (2) has a multiple factor $(p^2 + \alpha^2)^r$, we know from the theory of determinants that every first minor has a factor $(p^2 + \alpha^2)^{r-1}$. On evaluating the operational solution of (7) we can therefore cancel the factor $(p^2+\alpha^2)^{r-1}$ from the numerator and the denominator, and we are left with a single factor $(p^2+\alpha^2)$ in the denominator. Thus the part of the solution depending on the initial conditions is of the trigonometric form as usual; terms of the forms $t\cos\alpha t$ and $t\sin\alpha t$ do not arise.

1.8. In all the problems considered so far the operators that occur in the solutions are expansible in positive powers of Q, or in negative powers of p, and we have seen that so long as the series obtained by replacing Q by a number have a finite radius of convergence, however small, the operational solution is intelligible in terms of the definitions we have had. But we have not defined p as such, because we have only needed to define its negative powers, and p cannot be expanded in terms of its own negative powers. Then has p any meaning of its own? Since it replaces d/dx when we form the subsidiary equation we may naturally suppose that p means d/dx, and this is the meaning sometimes actually attributed to it; but care is needed. We recall that when the subsidiary equation is formed a term like py_0 appears on the right; but dy_0/dx is zero, so that if we pushed this interpretation too far we should be faced with the alarming result that the solutions of the equations do not depend on their initial values. The fact is that though the operators d/dx and Q both satisfy the laws of algebra and are freely commutative with constants, they are not as a rule commutative with each other. Thus

$$\frac{d}{dx}Qf(x)=\frac{d}{dx}\int_0^x f(x)\,dx=f(x), \quad\text{...............}(1)$$

$$Q\frac{d}{dx}f(x)=\int_0^x f'(x)\,dx=f(x)-f(0). \quad\text{.........}(2)$$

Thus the operators p and Q are commutative if, and only if, the function operated on vanishes with x. It appears from (1) that d/dx undoes the operation Q if d/dx acts after Q. With this convention we can identify p, the inverse of Q, with d/dx. When p and Q both occur in an operator, the Q operations must be carried out before the differentiations*.

This result explains why some of our interpretations differ from those given in text-books of differential equations for the determination of particular integrals. For instance, we have interpreted $1/(p-\alpha)$ as $\frac{1}{\alpha}(e^{\alpha x}-1)$. In the ordinary method, due to Boole, we expand this operator in ascending powers of p, and get

$$\frac{1}{p-\alpha}1=-\left(\frac{1}{\alpha}+\frac{p}{\alpha^2}+\dots\right)1=-\frac{1}{\alpha}. \quad\text{...............}(3)$$

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* Cf. Heaviside, *Electromagnetic Theory*, 2, 298.

This assumes that the sum of all the terms of the series after the first is zero. But

$$\left(\frac{1}{p-a}+\frac{1}{a}\right)1=\frac{p}{a\,(p-a)}\,1.\quad\text{................}(4)$$

If the operator on the right is written $\frac{1}{a}.\frac{1}{p-a}.p1$, it is clearly zero. But if it is written $\frac{1}{a}.p.\frac{1}{p-a}.1$, it gives by our rules $\frac{1}{a}e^{ax}$. The difference arises from the fact that the operators p and $(p-a)^{-1}$ are non-commutative.

We may notice, incidentally, that whereas the series in powers of Q give convergent series on interpretation, the same is not true of the corresponding series in powers of p. For instance, if $S=(x-1)^r$, where r is fractional, the series for $\frac{1}{p-1}\,.\,S$ in ascending powers of p diverges like $\Sigma\,(x-1)^{-n}n\,!$. Apart from the greater internal consistency of Heaviside's methods, then, they are capable of much wider application.

We have also

$$\begin{aligned} e^{hp}f(x) &= \left(1+hp+\frac{h^2p^2}{2\,!}+\dots\right)f(x) \\ &= f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\dots \\ &= f(x+h), \quad\text{..................................}(5) \end{aligned}$$

by Taylor's theorem. The operator e^{hp} increases the argument of a function by h.

This operator is useful in enabling us to find a theorem that plays in Heaviside's method a part closely analogous to that played by Fourier's integral theorem in the ordinary method. Instead of trigonometric functions, Heaviside treats as fundamental the function here called $H(x)$, which is zero for negative, and unity for positive, values of x*. This function being discontinuous, we cannot at once apply Taylor's theorem to it; but continuous functions can be found as near to it as we like†, and the results obtained by applying (5) may be regarded as the limits of results proved for these. Then we shall write

$$\left.\begin{aligned} e^{ph}H(x) &= 1 \text{ when } x>-h \\ &= 0 \text{ when } x<-h \end{aligned}\right\}. \quad\text{..................}(6)$$

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* *Proc. Roy. Soc.*, A, 52, 513.

† E.g. $\frac{1}{2}(\text{Erf}\,\lambda x+1)$ where λ may be made indefinitely great.

A fuller justification of this step will be given in the next chapter. Then if $h_1 < h_2$,

$$\left.\begin{aligned}(e^{-ph_1} - e^{-ph_2})\,H(x) &= 1 \text{ if } h_1 < x < h_2\\ &= 0 \text{ if } x < h_1\\ \text{and } &= 0 \text{ if } x > h_2\end{aligned}\right\}. \qquad \text{.............(7)}$$

If further $h_1 < h_2 < h_3 < \ldots < h_n$,

$$\{(e^{-ph_1} - e^{-ph_2}) f(h_1) + (e^{-ph_2} - e^{-ph_3}) f(h_2) + \ldots + (e^{-ph_{n-1}} - e^{-ph_n}) f(h_n)\}\, H(x)$$
$$= 0 \text{ for } x < h_1,\ f(h_1) \text{ for } h_1 < x < h_2, \text{ and so on.} \qquad \text{.........(8)}$$

If then the subdivisions of the interval h_1 to h_n become indefinitely numerous, we can approximate as closely as we like to a function $f(x)$ by a sum $\Sigma f(h)\,(e^{-ph_{r-1}} - e^{-ph_r})\,H(x)$. Now with a further proviso about replacing $H(x)$ by a continuous function and later proceeding to a limit, we can replace this sum by an integral. Then

$$f(x) = \int_{h=-\infty}^{\infty} f(h)\, d\,[e^{-ph}\, H(x)] = \int_{-\infty}^{\infty} f(h)\, e^{-ph}\, pH(x)\, dh. \quad (9)$$

If the integration with regard to h is carried out, we obtain a function of p operating on $H(x)$. In this way a given function of x can be expressed in the operational form*.

We notice that if h is positive

$$e^{-ph}\frac{1}{p}H(x) = e^{-ph}\begin{cases} 0 \text{ when } x < 0\\ x \text{ when } x > 0\end{cases}$$
$$= \begin{cases} 0 \text{ when } x < h\\ x - h \text{ when } x > h,\end{cases}$$

and
$$\frac{1}{p}e^{-ph}H(x) = \frac{1}{p}H(x-h) = \int_0^x H(x-h)\,dx$$
$$= 0 \text{ when } x < h,$$
$$= x - h \text{ when } x > h.$$

Thus the operators $\dfrac{1}{p}$ and e^{-ph} can be commuted provided h is positive.

But
$$e^{ph}\frac{1}{p}H(x) = \begin{cases} 0 \text{ when } x < -h\\ x + h \text{ when } x > -h,\end{cases}$$

and
$$\frac{1}{p}e^{ph}H(x) = \begin{cases} 0 \text{ when } x < 0\\ x \text{ when } x > 0.\end{cases}$$

Fortunately symbolic solutions seldom contain operators of the form e^{ph}, where h is positive, so that the fact that this operator is not commutative with $\dfrac{1}{p}$ gives little trouble in practice.

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* Heaviside, *Electromagnetic Theory*, 3, 327.

CHAPTER II

COMPLEX THEORY

2.1. All the interpretations found so far for the results of operations on unity or on $H(x)$ are special cases of two closely related general rules, as follows. If ϕ is an analytic function,

$$\phi(p).1 = \frac{1}{2\pi\iota}\int_C \frac{e^{\kappa x}}{\kappa}\phi(\kappa)\,d\kappa, \quad \text{..................(1)}$$

$$\phi(p)\,H(x) = \frac{1}{2\pi\iota}\int_L \frac{e^{\kappa x}}{\kappa}\phi(\kappa)\,d\kappa. \quad \text{..................(2)}$$

In the former definition the integration extends around a large circle in the complex plane. In the latter the integration is along a curve from $c-\iota\infty$ to $c+\iota\infty$, where c is positive and finite, such that all singularities of the integrand are on the left side of the path (that is, the side including $-\infty$). These rules are both due to Bromwich*.

Considering first (1), we have

$$p^{-n} = \frac{1}{2\pi\iota}\int_C \frac{e^{\kappa x}}{\kappa^{n+1}}\,d\kappa, \quad \text{........................(3)}$$

and the only pole of the integrand is at the origin, where the residue is $x^n/n\,!$. Hence when n is a positive integer

$$p^{-n} = \frac{x^n}{n\,!}. \quad \text{..............................(4)}$$

Also we see that

$$p^n = 0, \quad \text{.................................(5)}$$

since the integrand has no singularity in the finite part of the plane; and

$$p^0 = 1. \quad \text{.................................(6)}$$

The interpretation 1.5 (5) of $p/(p-\alpha)^n$ also follows immediately from (1).

The partial-fraction rule can also be proved easily. For by (1), with the notation of 1.5,

$$\frac{f(p)}{F(p)} = \frac{1}{2\pi\iota}\int_C \frac{e^{\kappa x}}{\kappa}\,\frac{f(\kappa)}{A(\kappa-\alpha_1)(\kappa-\alpha_2)\ldots(\kappa-\alpha_n)}\,d\kappa \quad \text{.........(7)}$$

$$= \frac{f(0)}{F(0)} + \Sigma\,\frac{f(\alpha)}{\alpha F'(\alpha)}\,e^{\alpha x} \quad \text{.............................(8)}$$

on evaluating the residues at the poles $0, \alpha_1, \alpha_2, \ldots \alpha_n$.

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* 'Normal Coordinates in Dynamical Systems,' *Proc. Lond. Math. Soc.*, 15 (1916), 401–448.

We can also prove that 1.4 (3), when interpreted by the rule (1), is the correct solution of equations 1.3 (1). If we denote by $e_{rs}(\kappa)$, $\Delta(\kappa)$ and $E_{rs}(\kappa)$ the results of replacing p by κ in e_{rs}, Δ, and E_{rs}, the part of 1.4 (3) not involving the S's is equivalent to

$$y_s = \frac{1}{2\pi\iota}\int_C \Sigma_r \frac{E_{rs}(\kappa)}{\Delta(\kappa)} v_r e^{\kappa x}\, d\kappa. \qquad \dots\dots(9)$$

Substituting in the differential equations 1.3 (1) we find that the left side of the m'th equation is

$$\frac{1}{2\pi\iota}\int_C \Sigma_r \Sigma_s \frac{E_{rs}(\kappa)}{\Delta(\kappa)} e_{ms}(\kappa)\, v_r e^{\kappa x}\, d\kappa. \qquad \dots\dots(10)$$

But

$$\begin{aligned} \Sigma_s E_{rs}(\kappa)\, e_{ms}(\kappa) &= \Delta(\kappa) \text{ if } r = m \\ &= 0 \text{ if } r \neq m, \end{aligned} \qquad \dots\dots(11)$$

and the expression (10) is then equal to

$$\frac{1}{2\pi\iota}\int_C v_m e^{\kappa x}\, d\kappa = 0. \qquad \dots\dots(12)$$

Thus the differential equations are satisfied.

Now considering the initial conditions, we put $x = 0$, and find

$$y_s = \frac{1}{2\pi\iota}\int_C \Sigma_r \frac{E_{rs}(\kappa)}{\Delta(\kappa)} v_r\, d\kappa. \qquad \dots\dots(13)$$

Also

$$v_r = \Sigma_m \frac{1}{\kappa}\{e_{rm}(\kappa) - b_{rm}\}\, u_m, \qquad \dots\dots(14)$$

and

$$\begin{aligned} y_s &= \frac{1}{2\pi\iota}\int_C \Sigma_r \Sigma_m \frac{E_{rs}(\kappa)}{\kappa\,\Delta(\kappa)}\{e_{rm}(\kappa) - b_{rm}\}\, u_m\, d\kappa \\ &= \frac{1}{2\pi\iota}\int \left\{\frac{1}{\kappa} u_s - \Sigma_r \Sigma_m \frac{E_{rs}(\kappa)}{\kappa\,\Delta(\kappa)} b_{rm} u_m\right\} d\kappa. \end{aligned} \qquad \dots\dots(15)$$

Now $\Delta(\kappa)$ is a polynomial of the n'th degree in κ, and $E_{rs}(\kappa)$ one of the $(n-1)$th degree. When $|\kappa|$ is great enough the second term in the integrand is of order κ^{-2}. It therefore gives zero on integration, and we have

$$y_s = u_s, \qquad \dots\dots(16)$$

so that the initial conditions are satisfied.

Next, suppose that instead of the S's being zero they are exponential terms of the form $Pe^{\mu x}$. Since the solutions are additive it is enough to suppose that the initial values of the y's are all zero, and to consider the effect of an exponential term in the first equation alone. Then the equations are

$$\left.\begin{aligned} &e_{11} y_1 + e_{12} y_2 + \dots + e_{1n} y_n = Pe^{\mu x} = \frac{Pp}{p-\mu} \\ &\Sigma_s e_{rs} y_s = 0 \quad (s \neq 1) \end{aligned}\right\}. \qquad \dots\dots(17)$$

The operational solution is

$$y_s = \frac{E_{1s}}{\Delta}\frac{Pp}{p-\mu}. \qquad \text{.........................(18)}$$

This is to be interpreted as

$$y_s = \frac{1}{2\pi\iota}\int_C \frac{E_{1s}(\kappa)}{\Delta(\kappa)}\frac{P}{\kappa-\mu}e^{\kappa x}\,d\kappa. \qquad \text{..............(19)}$$

Substituting in the differential equations we find that the left sides of all vanish except the first, in consequence of the integrand containing as a factor a determinant with two rows identical; the first gives

$$\frac{P}{2\pi\iota}\int_C \frac{e^{\kappa x}}{\kappa-\mu}\,d\kappa = Pe^{\mu x}. \qquad \text{.....................(20)}$$

The differential equations are therefore satisfied.

When $x=0$ and $|\kappa|$ is large, the integrands in (19) are all $O(\kappa^{-2})$ and the integrals are therefore zero. Thus all the y's vanish with x.

Since all functions that occur in physics can be expressed by Fourier's theorems as linear combinations of exponentials, the proof that the solution can be carried out by operational methods is complete. We shall see later, however, that Fourier's theorems are not so convenient to use as the formula 1.8 (9) based on the function $H(x)$.

2.2. Now let us consider the integrals 2.1 (2), taken along the path L. If we suppose the contour completed by a large semicircle from $c+\iota\infty$ to $c-\iota\infty$ by way of $-\infty$, all the singularities of the integrand are within the contour, and therefore the integral around it is the same as the integral (1) around a large circle. Now if x is positive, and $\phi(\kappa)/\kappa$ tends to zero uniformly with regard to arg κ as κ tends to ∞, the integral around a large semicircle tends to zero as the radius becomes indefinitely large, by Jordan's lemma*. Hence the integral along L is equivalent to the integral around a large circle. Thus if x is positive and

$$\phi(\kappa)/\kappa = O(\kappa^{-n})$$

when κ is great and n is positive, $\phi(p)H(x)$ is the same as $\phi(p)\,1$. But if x is negative, the integral around the large semicircle on the negative side of the imaginary axis is no longer an instance of Jordan's lemma, and this result no longer holds. The integral around a large semicircle on the *positive* side of the imaginary axis is, however, reducible to the integral along L, since by construction there is no singularity between these two paths. If then $\phi(\kappa)/\kappa = O(\kappa^{-n})$ when κ is great, the limit of the integral around this large semicircle is zero by Jordan's lemma, and therefore $\phi(p)H(x)$ is zero.

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* Whittaker and Watson, *Modern Analysis*, 1915, 115.

Thus if $\phi(p)$ is expansible in descending powers of p, beginning with a constant or a negative power of p, we shall have

$$\left.\begin{array}{ll}\phi(p)H(x)=\phi(p)1 & \text{when } x>0\\ \phi(p)H(x)=0 & \text{when } x<0\end{array}\right\}. \qquad \ldots\ldots\ldots\ldots(1)$$

The integrals 2.1 (2) for $p^n H(x)$, where n is a positive integer, are divergent, but these derivatives can be found by differentiation. Evidently they are zero except when $x=0$, when they do not exist.

Again, $$e^{ph}H(x)=\frac{1}{2\pi\iota}\int_L \frac{e^{\kappa(x+h)}}{\kappa}\,d\kappa=H(x+h). \qquad \ldots\ldots\ldots(2)$$

This proves the result obtained by less satisfactory means in 1.8.

We can also translate into the form of a double integral the operational expression for a general function. From 1.8 (9)

$$f(x)=\int_{-\infty}^{\infty} f(h)\,d\{e^{-ph}H(x)\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where the integration* gives a function of p operating on $H(x)$. Interpreting by Bromwich's rule

$$f(x)=\frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\int_L f(h)\,e^{\kappa(x-h)}\,dh\,d\kappa, \qquad \ldots\ldots\ldots\ldots(4)$$

where the integration with regard to h is to be carried out first.

If the function of p that arises when (3) is integrated has no singularity on the imaginary axis or on the positive side of it, and if further it contains p as a factor, we can replace the integration with regard to κ in (4) by one along the imaginary axis. Then we can put

$$\kappa=\iota\lambda, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and we have

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(h)\sin\lambda(x-h)\,dh\,d\lambda. \qquad \ldots\ldots\ldots\ldots(6)$$

This is Fourier's integral theorem.

2.3. In practice the form 2.1 (2) is generally used in preference to 2.1 (1). In problems involving a finite number of ordinary differential equations of the first order the forms are equivalent for all positive values of the independent variable; and since the independent variable is usually the time, and we require to know how a system will behave if it is in a given state at time 0 and is afterwards acted upon by known disturbances, it is as a rule only positive values of the time that concern us. But when we come to deal with continuous systems it usually happens that when the operational solution $\phi(p)$ is interpreted as an integral, the integrand has an infinite number of poles, and that no circle however large can include all of them. Hence the contour integral 2.1 (1) cannot be formed. But the line integral 2.1 (2) usually still exists. Again, it may happen that $\phi(\kappa)$ has a branch-point at the origin or on the negative side of the imaginary axis. Here again the contour integral does not exist, but the line integral does.

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* The integral is of the type introduced by Stieltjes.

Consequently in most of the writings of Heaviside and Bromwich, when a function of an operator $\phi(p)$ occurs without the operand being stated explicitly, it is to be understood that the operand is $H(x)$, and that the interpretation 2.1 (2) is to be adopted. This rule will be followed in this work when we come to treat continuous systems; and if it is also supposed to be adopted in the problems of the next chapter no harm will be done.

Considerable latitude is admissible in choosing the path L. Since by construction the integrand is regular at all points to the right of L, we may replace L by any other path on its positive side provided it ends at $c \pm \iota\infty$. Again, the systems we are to discuss are usually stable systems; that is, the poles of $\phi(\kappa)$ are all on the imaginary axis or to the left of it. Then L can be taken as near as we like to the imaginary axis. L must not cross the imaginary axis if there are poles on the latter; but it can be taken to be a line parallel to the imaginary axis and distant c from it. This is the device most often used by Bromwich; the only advantage of the present definition of L is that the solution is then adaptable to unstable systems as well as stable ones. In all cases the actual value of c does not affect the results, so long as it is positive.

2.4. The principal operators involving branch-points are fractional powers of p. We require then an interpretation of $p^n H(x)$, where n is fractional*. By our rule

$$p^n H(x) = \frac{1}{2\pi\iota}\int_L \kappa^{n-1} e^{\kappa x}\, d\kappa, \quad \dots\dots\dots\dots\dots\dots(1)$$

and the integral converges if $n < 1$. A contour including L as part of itself, and such that the integrand is regular within it, is as shown. Evidently if x is positive the large quadrants make no contribution.

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* Cf. Bromwich, *Proc. Camb. Phil. Soc.*, 20 (1921), 411–427.

The integral along L (= AB) is therefore in the limit equal and opposite to one along $CDEF$. If further n is positive the contribution from DE tends to zero with the radius of DE. Thus AB, CD and EF contribute the whole of the integral. Now on CD and EF

$$\kappa = \mu\, e^{\iota\pi} \text{ and } \mu\, e^{-\iota\pi}. \quad\ldots\ldots\ldots\ldots(2)$$

respectively, where μ is real and positive. Hence on CD

$$\left.\begin{aligned} &\kappa^{n-1}\, e^{\kappa x}\, d\kappa = -\,\mu^{n-1}\, e^{-\mu x}\, e^{\iota\pi\,(n-1)}\, d\mu \\ \text{and on } EF \qquad &\phantom{\kappa^{n-1}\, e^{\kappa x}\, d\kappa} = -\,\mu^{n-1}\, e^{-\mu x}\, e^{-\iota\pi\,(n-1)}\, d\mu \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(3)$$

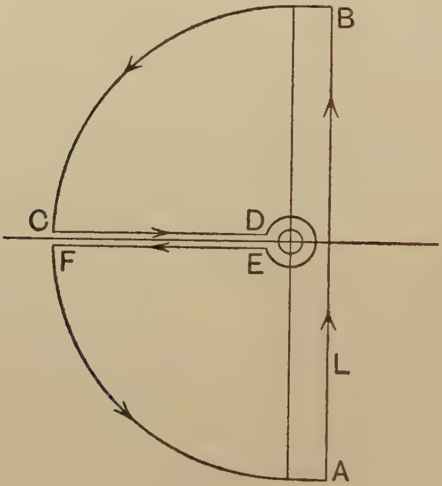

Fig. 1.

Therefore

$$\begin{aligned} p^n H(x) &= -\frac{1}{\pi}\int_0^\infty \mu^{n-1}\, e^{-\mu x}\, d\mu \sin(n-1)\pi \\ &= \frac{\sin n\pi}{\pi}\,\frac{\Gamma(n)}{x^n}. \quad\ldots\ldots\ldots\ldots(4) \end{aligned}$$

But by a known theorem

$$\Gamma(n)\,\Gamma(1-n) = \pi \operatorname{cosec} n\pi, \quad\ldots\ldots\ldots\ldots(5)$$

and therefore

$$p^n H(x) = \frac{x^{-n}}{\Gamma(1-n)}. \quad\ldots\ldots\ldots\ldots(6)$$

This result has been obtained for $0 < n < 1$. If we change n into $-m$, where $-1 < m < 0$, we have

$$p^{-m} H(x) = \frac{x^m}{\Gamma(m+1)}. \quad \text{.....................}(7)$$

Powers of p outside this range can be found now by integration or differentiation. We have

$$p^{-m+1} H(x) = p \cdot \frac{x^m}{\Gamma(m+1)} = \frac{x^{m-1}}{\Gamma(m)}, \quad \text{...........}(8)$$

which shows that (7) still holds when $m-1$ is written for m. (To justify this, the path AB must be replaced by $FEDC$.)

Similarly

$$p^{-m-1} H(x) = \int_0^x \frac{x^m}{\Gamma(m+1)}\, dx = \frac{x^{m+1}}{\Gamma(m+2)}. \quad \text{.........}(9)$$

By induction we may therefore generalize (7) to any value of m. Hence

$$p^{-m} H(x) = \frac{x^m}{\Gamma(m+1)}. \quad \text{.....................}(10)$$

Since when m is a positive integer $\Gamma(m+1) = m\,!$, and when m is a negative integer $\Gamma(m+1)$ is infinite, the interpretations already adopted for integral powers of p are special cases of these.

In particular, since

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \text{........................}(11)$$

we have

$$p^{\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}; \; p^{\frac{3}{2}} = -\frac{1}{2\sqrt{\pi}} x^{-\frac{3}{2}}; \; p^{\frac{5}{2}} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{\sqrt{\pi}} x^{-\frac{5}{2}}; \quad \text{......}(12)$$

and so on. Also

$$p^{-\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}. \quad \text{........................}(13)$$

A related function that arises in problems of heat conduction is e^{-aq}, where a is a constant, and q denotes $p^{\frac{1}{2}}$. By Bromwich's rule

$$e^{-aq} H(x) = \frac{1}{2\pi\iota} \int_L \frac{e^{\kappa x - a\kappa^{\frac{1}{2}}}}{\kappa}\, d\kappa. \quad \text{...............}(14)$$

On L the argument of $\kappa^{\frac{1}{2}}$ is between $\pm \frac{1}{4}\pi$. Hence if a is positive the integral is convergent.

Immediate expansion of $e^{-a\kappa^{\frac{1}{2}}}$ in a power series and integration term by term would not be legitimate, because all the resulting integrals after the first two would diverge. But (14) is equivalent to an integral along a path such as *FEDC* in Fig. 1, and on this path we can proceed in this way. Thus

$$e^{-qa}H(x) = \frac{1}{2\pi\iota}\int_F^C \frac{e^{\kappa x}}{\kappa}\left[1 - a\kappa^{\frac{1}{2}} + \frac{a^2\kappa}{2!} - \frac{a^3\kappa^{\frac{3}{2}}}{3!} + \dots\right] d\kappa. \quad \dots(15)$$

All the positive integral powers of κ give zero on integration. The other terms are equivalent to

$$1 - ap^{\frac{1}{2}} - \frac{a^3p^{\frac{3}{2}}}{3!} - \frac{a^5p^{\frac{5}{2}}}{5!} \dots = 1 - \frac{2}{\sqrt{\pi}}\left\{\frac{a}{2\sqrt{x}} - \frac{1}{3}\left(\frac{a}{2\sqrt{x}}\right)^3 + \frac{1}{2!5}\left(\frac{a}{2\sqrt{x}}\right)^5 \dots\right\}$$

$$= 1 - \operatorname{Erf}\frac{a}{2\sqrt{x}}, \quad \dots\dots(16)$$

where, by definition,

$$\operatorname{Erf} w = \frac{2}{\sqrt{\pi}}\int_0^w e^{-t^2}\,dt. \quad \dots\dots\dots\dots\dots\dots\dots(17)$$

By differentiation with regard to a we find

$$qe^{-qa}H(x) = \frac{1}{\sqrt{(\pi x)}}e^{-a^2/4x}. \quad \dots\dots\dots\dots\dots\dots(18)$$

We shall sometimes need an asymptotic approximation to Erf w when w is great. We have

$$1 - \operatorname{Erf} w = \frac{2}{\sqrt{\pi}}\int_w^\infty e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\int_{w^2}^\infty e^{-u}u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{\sqrt{\pi}}e^{-w^2}w^{-1}\left[1 - \frac{1}{2}w^{-2} + \frac{1.3}{2.2}w^{-4} - \frac{1.3.5}{2.2.2}w^{-6} + \dots\right]$$

$$+ \frac{(-1)^n}{\sqrt{\pi}}\frac{1.3.5\dots(2n-1)}{2.2.2\dots2}\int_{w^2}^\infty e^{-u}u^{-\frac{1}{2}(2n+1)}\,du, \quad (19)$$

on successive integrations by parts. But the last integral

$$= 2\int_w^\infty e^{-t^2}t^{-2n}\,dt < 2e^{-w^2}\int_w^\infty t^{-2n}\,dt,$$

$$< \frac{2e^{-w^2}}{2n-1}w^{-(2n-1)}, \quad \dots\dots\dots\dots(20)$$

so that its contribution to the function is less than the previous term. We have therefore the asymptotic approximation

$$e^{w^2}(1 - \operatorname{Erf} w) \sim \frac{1}{\sqrt{\pi}}\left[w^{-1} - \frac{1}{2}w^{-3} + \frac{1.3}{2.2}w^{-5} - \frac{1.3.5}{2.2.2}w^{-7} + \dots\right]. \quad (21)$$

CHAPTER III

PHYSICAL APPLICATIONS: ONE INDEPENDENT VARIABLE

3.1. An electric circuit contains a cell, a condenser, and a coil with self-induction and resistance. Initially the circuit is open. It is suddenly completed; find how the charge on the plates varies with the time.

Let y be the charge on the plates, t the time, K the capacity of the condenser, L the self-induction, and R the resistance of the circuit; and let E be the electromotive force of the cell. Write σ for d/dt.

The current in the circuit is $\dot{y}$, and the charging of the plates of the condenser produces a potential difference y/K tending to oppose the original E.M.F. Then y satisfies the differential equation

$$E-\frac{y}{K}=L\ddot{y}+R\dot{y}. \qquad \text{.......................(1)}$$

Initially y and $\dot{y}$, the current, are zero. Hence the subsidiary equation is simply

$$\left(L\sigma^2+R\sigma+\frac{1}{K}\right)y=E, \qquad \text{....................(2)}$$

and the solution is

$$y=\frac{E}{L\sigma^2+R\sigma+\frac{1}{K}}. \qquad \text{.......................(3)}$$

If now the denominator is expressed in the form $L(\sigma+\alpha)(\sigma+\beta)$, the interpretation is, by 1.5 (9)

$$y=\frac{E}{L}\left[KL+\frac{1}{\alpha-\beta}\left(\frac{1}{\alpha}e^{-\alpha t}-\frac{1}{\beta}e^{-\beta t}\right)\right]. \qquad \text{...........(4)}$$

Since $\alpha+\beta$ and $\alpha\beta$ are both positive, α and β must either be both real and positive, or conjugate imaginaries with positive real parts. In either case y tends to KE as a limit, as we should expect.

We notice incidentally that if the circuit contained no capacity or self-induction the differential equation would be simply

$$R\sigma y=E. \qquad \text{...............................(5)}$$

Hence if a problem has been solved for simple resistances, self-induction and capacity can be allowed for by writing $L\sigma+R+\frac{1}{K\sigma}$ for R. For this reason this expression is sometimes called a 'resistance operator,' and the method generally the 'method of resistance operators.'

3.2. *The Wheatstone Bridge method of determining Self-Induction.* In this method the unknown inductance is placed in the first arm of a Wheatstone bridge; the fourth arm is shunted, a known capacity being placed in the shunt*.

First consider the ordinary Wheatstone bridge, the resistances of the arms being R_1, R_2, R_3, R_4; let x be the current in R_1, y that in R_2, g that through the galvanometer, and b the resistance of the battery and leads.

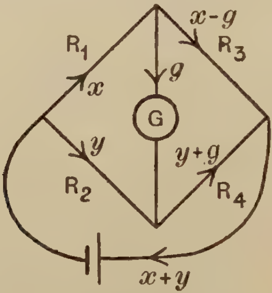

Fig. 2.

Then
$$R_1 x - R_2 y + Gg = 0, \quad \text{..................(1)}$$
$$R_3 x - R_4 y - (R_3 + R_4 + G)\,g = 0, \quad \text{..................(2)}$$
$$b\,(x+y) + R_2 y + R_4 (y+g) = E, \quad \text{..................(3)}$$
and on solving we find
$$\frac{g}{R_2 R_3 - R_1 R_4} = \frac{x+y}{G(R_1 + R_2 + R_3 + R_4) + (R_1 + R_2)(R_3 + R_4)}. \quad (4)$$
If b is large compared with R_2 and R_4, we have nearly
$$x + y = E/b, \quad \text{..............................(5)}$$
and
$$g = \frac{(E/b)(R_2 R_3 - R_1 R_4)}{G(R_1 + R_2 + R_3 + R_4) + (R_1 + R_2)(R_3 + R_4)}. \quad \text{......(6)}$$
According to the result of 3.1, we can allow for the self-induction in the first arm by replacing R_1 by $L\sigma + R_1$. Let the arrangement in the fourth arm be as shown (Fig. 3). The resistance of the main wire is R_4,

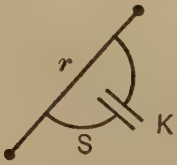

Fig. 3.

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* Bromwich, *Phil. Mag.* 37 (1919), 407–419. Further references are given.

that of the shunted portion of it r. Suppose the shunt to have a resistance S. Then the effective resistance of the whole arm would be

$$R_4 - r + \frac{rS}{r+S} = R_4 - \frac{r^2}{r+S}. \qquad \text{.................}(7)$$

If the shunt contains a capacity K, we must replace S by $S + 1/K\sigma$. Hence in the formula for g we must replace R_1 by $L\sigma + R_1$ and R_4 by

$$R_4 - \frac{r^2 K\sigma}{(r+S)\,K\sigma + 1}.$$

The result expresses the current through the galvanometer when the battery circuit is suddenly closed.

It can be shown that in actual conditions g cannot vanish for all values of the time. A sufficient condition for this would be that the operator $R_2R_3 - R_1R_4$ should be identically zero; then g would be identically zero whatever the remaining factor might represent. But with our modifications this factor becomes

$$R_2 R_3 - (L\sigma + R_1)\left(R_4 - \frac{r^2 K\sigma}{(r+S)K\sigma + 1}\right). \qquad \text{...........}(8)$$

Multiplying up and equating coefficients of powers of σ to zero, we find

$$R_4(r+S) = r^2, \qquad \text{...........................}(9)$$

$$-LR_4 + (R_2R_3 - R_1R_4)(r+S)K + R_1 r^2 K = 0, \qquad \text{........}(10)$$

$$R_2R_3 - R_1R_4 = 0. \qquad \text{.........................}(11)$$

From the construction of the apparatus $r \leqslant R_4$, $r \leqslant r + S$. Thus (9) can hold only if $r = R_4$ and $S = 0$. The shunt wire must be attached to the ends of R_4 and must have zero resistance. Substituting in (10) we find

$$L = R_1 R_4 K, \qquad \text{........................}(12)$$

together with the usual condition (11) for permanent balance of the bridge.

Actually these conditions for complete balance cannot be completely satisfied; but for the determination of L it is not necessary that they should. Suppose the galvanometer is a ballistic one, and that it is so adjusted that there are no permanent current and no throw on closing the circuit. Thus

$$\lim_{t\to\infty} g = 0; \int_0^\infty g\,dt = 0. \qquad \text{....................}(13)$$

If g is expressed in the form $f(\sigma)/F(\sigma)$, we have

$$g = \frac{f(0)}{F(0)} + \Sigma \frac{f(\alpha)}{\alpha F(\alpha)} e^{\alpha t}, \qquad \text{.....................}(14)$$

where all the α's, in the conditions of the problem, will have negative real parts. Then the condition that g shall tend to zero gives

$$f(0) = 0. \qquad\qquad (15)$$

Also
$$\int_0^\infty g\,dt = \Sigma \frac{f(\alpha)}{\alpha^2 F'(\alpha)}. \qquad\qquad (16)$$

Equation (15) shows that the operational form of g contains σ as a factor. Also we can write

$$g = \Sigma \frac{f(\alpha)}{\alpha F'(\alpha)} \frac{\sigma}{\sigma - \alpha} \qquad\qquad (17)$$

so that (16) is the limit of $(-g/\sigma)$ when σ tends to zero. The vanishing of Lim g and $\int_0^\infty g\,dt$ imply therefore that the operational form of g contains σ^2 as a factor. Hence the modified form of $R_2R_3 - R_1R_4$ must contain σ^2 as a factor. Then (10) and (11) still hold; (9) no longer holds. We now have

$$R_2R_3 - R_1R_4 = 0, \qquad\qquad (18)$$

$$LR_4 = R_1 r^2 K, \qquad\qquad (19)$$

which gives the required rule for finding L.

3.3. *The Seismograph.* In principle most seismographs are Euler pendulums—pendulums with supports rigidly attached to the earth, so that when the earth's surface moves it displaces the point of support horizontally and disturbs the pendulum. The seismograph differs from the Euler pendulum as considered in text-books of dynamics in two ways: instead of being free to vibrate in a vertical plane, it is constrained to swing, like a gate, about an axis nearly, but not quite, vertical, so that the period is much lengthened; and fluid viscosity or electromagnetic damping is introduced to give a frictional term proportional to the velocity. The displacement of the mass with regard to the earth then satisfies an equation of the form

$$\ddot{x} + 2\kappa\dot{x} + n^2 x = \lambda\ddot{\xi}, \qquad\qquad (1)$$

where ξ is the displacement of the ground, and κ, n, and λ are constants of the instrument*. Suppose first that the ground suddenly acquires a finite velocity, say unity. Then

$$\dot{\xi} = H(t) \qquad\qquad (2)$$

---

* Some instruments, such as that of Wiechert, are not on the principle of the Euler pendulum, but nevertheless give an equation of this form.

and our subsidiary equation is

$$(\sigma^2+2\kappa\sigma+n^2)\,x=\lambda\sigma H(t). \qquad \text{.........(3)}$$

Put $$\sigma^2+2\kappa\sigma+n^2=(\sigma+\alpha)(\alpha+\beta). \qquad \text{.........(4)}$$

Then $$x=\frac{\lambda\sigma}{(\sigma+\alpha)(\sigma+\beta)}H(t) \qquad \text{.........(5)}$$

$$=0 \text{ when } t<0 \qquad \text{.........(6)}$$

and $$=\frac{\lambda}{\alpha-\beta}(e^{-\beta t}-e^{-\alpha t}) \text{ when } t>0. \qquad \text{.........(7)}$$

The recorded displacement x therefore begins by increasing at a finite rate λ, reaches a maximum $\lambda\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\alpha-\beta}}$ after a time $\frac{1}{\alpha-\beta}\log\frac{\alpha}{\beta}$, and then tends asymptotically to zero.

If α and β are real, and β less than α, we see that the behaviour after a long time depends mainly on $e^{-\beta t}$; now as the experimental ideal is to confine the effects of a disturbance to as short an interval afterwards as possible, we see that we should make β as large as possible. But

$$\beta=\kappa-(\kappa^2-n^2)^{\frac{1}{2}}=\frac{n^2}{\kappa+(\kappa^2-n^2)^{\frac{1}{2}}}, \qquad \text{.........(8)}$$

and for a given n, β is greatest when $\kappa=n$. This is the condition for what is called aperiodicity; the roots of the period equation are equal, real, and negative. Many seismographs are arranged so as to satisfy this condition. The solution is then

$$x=\frac{\lambda\sigma}{(\sigma+n)^2}H(t) \qquad \text{.........(9)}$$

$$=0 \text{ when } t<0 \qquad \text{.........(10)}$$

$$=\lambda t e^{-nt} \text{ when } t>0. \qquad \text{.........(11)}$$

The maximum displacement is now at time $1/n$ after the start, and is equal to λ/en.

If $\kappa<n$, we can put

$$n^2-\kappa^2=\gamma^2. \qquad \text{.........(12)}$$

Then the solution is

$$x=\frac{\lambda}{\gamma}e^{-\kappa t}\sin\gamma t, \qquad \text{.........(13)}$$

and the motion does not die down so rapidly as for $\kappa = n$. The aperiodic state therefore gives the least motion after a long interval for a given value of n.

In practice, however, κ is usually made rather less than n; in the Milne-Shaw machine, for instance, κ is about $0{\cdot}7n$. The motion is then oscillatory, but the ratio of the first swing to the second is $e^{\kappa\pi/\gamma}$, or about 20. But x vanishes after an interval π/γ from the start, or about $4/n$, and ever afterwards is a small fraction of its first maximum. The reduced damping effect after a long time is considered less important than the quick and complete recovery after the first maximum. The time of the first maximum is $1{\cdot}1/n$ from the start, as against $1/n$ for the aperiodic instrument and $1{\cdot}57/n$ for the undamped one.

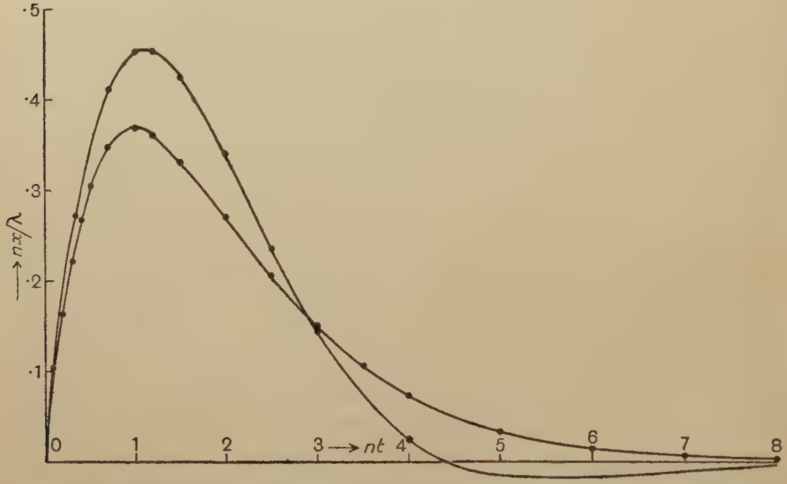

Fig. 4. Recovery of seismographs with $\kappa = n$ and $\kappa = n/\sqrt{2}$ after same impulse.

3.31. The Galitzin seismograph is similarly arranged, but the motion of the pendulum generates by electromagnetic induction a current, which passes through a galvanometer. If x is the displacement of the pendulum, and y that of the galvanometer, the differential equations are

$$\ddot{x} + 2\kappa_1\dot{x} + n_1^2 x = \lambda\ddot{\xi}, \quad\text{.........................(1)}$$

$$\ddot{y} + 2\kappa_2\dot{y} + n_2^2 y = \mu\dot{x}. \quad\text{.........................(2)}$$

Supposing both pendulum and galvanometer to start from rest, we have

$$y = \frac{\lambda\mu\sigma^3\xi}{(\sigma^2 + 2\kappa_1\sigma + n_1^2)(\sigma^2 + 2\kappa_2\sigma + n_2^2)}. \quad\text{...............(3)}$$

As a rule the two interacting systems are so arranged that κ and n are the same for both, and both are aperiodic, so that $\kappa = n$. Then

$$y = \frac{\lambda\mu\sigma^3\xi}{(\sigma+n)^4}. \qquad\text{.............................(4)}$$

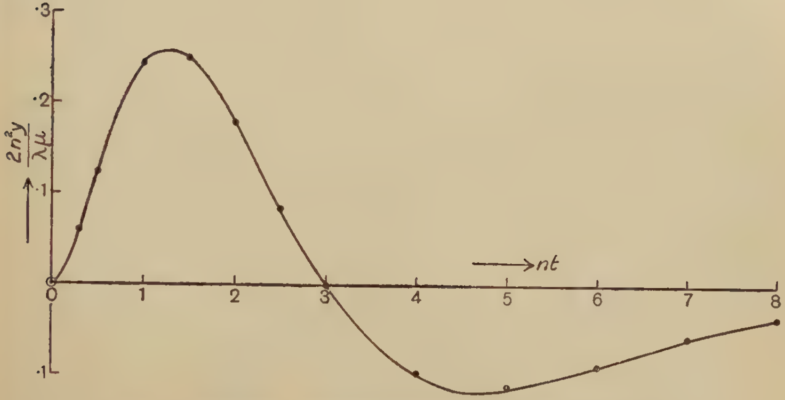

Fig. 5. Recovery of Galitzin seismograph after impulse.

If we suppose as before that

$$\dot{\xi} = H(t), \qquad\text{.............................(5)}$$

we have
$$y = \frac{\lambda\mu\sigma^2}{(\sigma+n)^4} H(t),$$
$$= \lambda\mu \frac{d}{dt}\left(\frac{t^3}{3\,!} e^{-nt}\right) \text{ when } t > 0,$$
$$= \tfrac{1}{2}\lambda\mu\,(t^2 - \tfrac{1}{3}nt^3)\,e^{-nt}. \qquad\text{..........................(6)}$$

The indicator therefore begins to move with a finite acceleration, instead of with a finite velocity as for the pendulum. The maximum displacement follows after time $(3-\sqrt{3})/n = 1{\cdot}27/n$, the mirror passes through the equilibrium position after time $3/n$, and there is a maximum displacement in the opposite direction after time $4{\cdot}73/n$. The mirror then returns asymptotically to the position of equilibrium. The ratio of the two maxima is $e^{2\sqrt{3}}/(2+\sqrt{3})^2 = 2{\cdot}3$. In comparison with an instrument such as the Milne-Shaw, recording the displacement of the pendulum directly, the Galitzin machine gives the first maximum a little later, the first zero a little earlier, and the next maximum displacement is larger in comparison with the first maximum.

In an actual earthquake the velocity of the ground is annulled by other waves arriving later; the complete motion of the seismograph is a combination of those given by the separate displacements of the ground.

3.4. *Resonance.* A simple pendulum, originally hanging in equilibrium, is disturbed for a finite time by a force varying harmonically in a period equal to the free period of the pendulum. Find the motion after the force is removed.

The differential equation is

$$\ddot{x} + n^2 x = f \sin nt, \tag{1}$$

$$= f \frac{n\sigma}{\sigma^2 + n^2}. \tag{2}$$

The solution is then

$$x = f \frac{n\sigma}{(\sigma^2 + n^2)^2}, \tag{3}$$

nothing having to be added on account of the initial conditions.

To evaluate this, we notice that

$$\frac{\sigma}{\sigma^2 + n^2} = \frac{1}{n} \sin nt. \tag{4}$$

Differentiating with regard to n, we have

$$-\frac{2n\sigma}{(\sigma^2 + n^2)^2} = \frac{1}{n} t \cos nt - \frac{1}{n^2} \sin nt, \tag{5}$$

and

$$x = \frac{f}{2n^2} (\sin nt - nt \cos nt). \tag{6}$$

Suppose the disturbance acts for time $r\pi/n$, where r is an integer. At the end of this time

$$x = -\frac{f}{2n^2} r\pi (-1)^r; \quad \dot{x} = 0. \tag{7}$$

The subsequent motion is therefore given by

$$x = -\frac{(-1)^r r\pi f}{2n^2} \cos (nt - r\pi),$$

$$= -\frac{r\pi f}{2n^2} \cos nt. \tag{8}$$

✓ **3.5.** Three particles of masses m, $\frac{21}{20}m$, and m, in order, are attached to a light stretched string, the ends of the string being fixed. One of the particles of mass m is struck by a transverse impulse I. Find the subsequent motion of the middle particle. (Intercollegiate Examination, 1923.)

If x_1, x_2, x_3 are the displacements of the three particles, P the tension, and l the distance between consecutive particles, we find in the usual way the three equations of motion

$$\ddot{x}_1 = -\lambda(2x_1 - x_2), \quad \text{.......................}(1)$$

$$\tfrac{21}{20}\ddot{x}_2 = -\lambda(-x_1 + 2x_2 - x_3), \quad \text{..................}(2)$$

$$\ddot{x}_3 = -\lambda(-x_2 + 2x_3), \quad \text{.......................}(3)$$

where

$$\lambda = \frac{P}{ml}. \quad \text{..........................}(4)$$

Initially all the displacements are zero, $\dot{x}_2 = \dot{x}_3 = 0$, $\dot{x}_1 = I/m$. Hence the subsidiary equations are

$$(\sigma^2 + 2\lambda)\,x_1 - \lambda x_2 = \sigma I/m, \quad \text{..............}(5)$$

$$-\lambda x_1 + (\tfrac{21}{20}\sigma^2 + 2\lambda)\,x_2 - \lambda x_3 = 0, \quad \text{..................}(6)$$

$$-\lambda x_2 + (\sigma^2 + 2\lambda)\,x_3 = 0. \quad \text{..................}(7)$$

Hence

$$x_3 = \frac{\lambda}{\sigma^2 + 2\lambda}\,x_2; \quad x_1 = \frac{\lambda}{\sigma^2 + 2\lambda}\,x_2 + \frac{\sigma}{\sigma^2 + 2\lambda}\,\frac{I}{m}, \quad \text{........}(8)$$

and

$$\left(\frac{21}{20}\,\sigma^2 + 2\lambda - \frac{2\lambda^2}{\sigma^2 + 2\lambda}\right) x_2 = \frac{\lambda\sigma}{\sigma^2 + 2\lambda}\,\frac{I}{m}. \quad \text{..............}(9)$$

Multiplying by $\sigma^2 + 2\lambda$ we find

$$(7\sigma^2 + 4\lambda)(3\sigma^2 + 10\lambda)\,x_2 = 20\lambda\sigma I/m, \quad \text{...........}(10)$$

and

$$\begin{aligned} x_2 &= \frac{20\lambda\sigma}{(7\sigma^2 + 4\lambda)(3\sigma^2 + 10\lambda)}\,\frac{I}{m} \\ &= \frac{20}{58}\,\frac{I}{m}\left\{\frac{7\sigma}{7\sigma^2 + 4\lambda} - \frac{3\sigma}{3\sigma^2 + 10\lambda}\right\} \\ &= \frac{10}{29}\,\frac{I}{m}\left\{\frac{\sin\alpha t}{\alpha} - \frac{\sin\beta t}{\beta}\right\}, \quad \text{....................}(11) \end{aligned}$$

where

$$\alpha^2 = \tfrac{4}{7}\lambda; \quad \beta^2 = \tfrac{10}{3}\lambda. \quad \text{.......................}(12)$$

We notice that the mode of speed $\sqrt{(2\lambda)}$ is excited, but does not need to be evaluated because it does not affect the middle particle.

3.6. *Radioactive Disintegration of Uranium.* The uranium family of elements are such that an atom of any one of them, except the last, is capable of breaking up into an atom of the next and an atom of helium*. The helium atom undergoes no further change. The number of atoms of any element in a given specimen that break up in a short interval of time is proportional to the time interval and to the number of atoms of that element present. If then u, x_1, x_2, ... x_n are the numbers of atoms of the various elements present at time t, they will satisfy the differential equations

---

* We neglect β-ray products, for reasons that will appear later.

$$\left.\begin{aligned}
\frac{du}{dt} &= -\kappa u,\\
\frac{dx_1}{dt} &= \kappa u - \kappa_1 x_1,\\
\frac{dx_2}{dt} &= \kappa_1 x_1 - \kappa_2 x_2,\\
&\cdots\cdots\cdots\cdots\\
\frac{dx_n}{dt} &= \kappa_{n-1} x_{n-1}.
\end{aligned}\right\}\cdots\cdots\cdots\cdots(1)$$

Suppose that initially only uranium is present. Thus when $t=0$, $u=u_0$, and all the other dependent variables are zero. Then the subsidiary equations are

$$\left.\begin{aligned}
(\sigma+\kappa)\,u &= \sigma u_0,\\
(\sigma+\kappa_1)\,x_1 &= \kappa u,\\
(\sigma+\kappa_2)\,x_2 &= \kappa_1 x_1,\\
&\cdots\cdots\cdots\cdots\\
(\sigma+\kappa_{n-1})\,x_{n-1} &= \kappa_{n-2}\,x_{n-2},\\
\sigma x_n &= \kappa_{n-1} x_{n-1}.
\end{aligned}\right\}\cdots\cdots\cdots\cdots(2)$$

The operational solutions are

$$\left.\begin{aligned}
u &= \frac{\sigma u_0}{\sigma+\kappa};\quad x_1 = \frac{\kappa\sigma u_0}{(\sigma+\kappa)(\sigma+\kappa_1)};\\
x_2 &= \frac{\kappa\kappa_1\sigma u_0}{(\sigma+\kappa)(\sigma+\kappa_1)(\sigma+\kappa_2)};\\
&\cdots\cdots\cdots\cdots\\
x_n &= \frac{\kappa\kappa_1\ldots\kappa_{n-1}u_0}{(\sigma+\kappa)(\sigma+\kappa_1)\ldots(\sigma+\kappa_{n-1})}.
\end{aligned}\right\}\cdots\cdots\cdots\cdots(3)$$

These are directly adapted for interpretation by the partial-fraction rule. In fact

$$\left.\begin{aligned}
u &= u_0 e^{-\kappa t};\quad x_1 = \frac{\kappa u_0}{\kappa_1-\kappa}\left(e^{-\kappa t}-e^{-\kappa_1 t}\right);\\
x_2 &= \kappa\kappa_1 u_0\left\{\frac{1}{(\kappa_1-\kappa)(\kappa_2-\kappa)}e^{-\kappa t}+\frac{1}{(\kappa-\kappa_1)(\kappa_2-\kappa_1)}e^{-\kappa_1 t}\right.\\
&\qquad\qquad\left.+\frac{1}{(\kappa-\kappa_2)(\kappa_1-\kappa_2)}e^{-\kappa_2 t}\right\};\\
x_n &= u_0 - \frac{\kappa_1\ldots\kappa_{n-1}u_0}{(\kappa_1-\kappa)\ldots(\kappa_{n-1}-\kappa)}e^{-\kappa t}-\ldots.
\end{aligned}\right\}\cdots(4)$$

Of all the decay constants κ is much the smallest. If the time elapsed is long enough for all the exponential functions except $e^{-\kappa t}$ to have become insignificant, these results reduce approximately to

$$u = u_0 e^{-\kappa t}; \; x_1 = \frac{\kappa}{\kappa_1} e^{-\kappa t}; \quad x_2 = \frac{\kappa}{\kappa_2} e^{-\kappa t}; \; \ldots \; x_n = u_0 (1 - e^{-\kappa t}). \quad \ldots\ldots(5)$$

With the exception of the last, the quantities of the various elements decrease, retaining constant ratios to one another.

On the other hand, if the time elapsed is so short that unity is still a first approximation to all the exponential functions, we can proceed by expanding the operators in descending powers of σ and interpreting term by term. Hence we see that at first x_1 will increase in proportion to t, x_2 to t^2, and x_n to t^n.

In experimental work an intermediate condition often occurs. Some of the exponentials may become insignificant in the time occupied by an experiment, while others are still nearly unity. We have

$$x_r = \frac{\kappa_{r-1} x_{r-1}}{\sigma + \kappa_r} = \kappa_{r-1} [\sigma^{-1} x_{r-1} - \kappa_r \sigma^{-2} x_{r-1} + \ldots] \quad \ldots\ldots\ldots(6)$$

and if $\kappa_r t$ is small we can neglect the second and later terms in comparison with the first. Thus in this case

$$x_r = \kappa_{r-1} \sigma^{-1} x_{r-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

If x_{r-1} is of the form t^s, we can put

$$x_r = \frac{\kappa_{r-1} s!}{(\sigma + \kappa_r)\sigma^s} = \frac{\kappa_{r-1} s!}{\sigma^{s+1}} \frac{\sigma}{\sigma + \kappa_r} = \frac{\kappa_{r-1} s!}{\sigma^{s+1}} e^{-\kappa_r t}. \quad \ldots\ldots\ldots(8)$$

If $\kappa_r t$ is small, we can replace the exponential by unity and confirm (7). If it is great,

$$\sigma^{-1} e^{-\kappa_r t} = \int_0^t e^{-\kappa_r t} \, dt = \frac{1}{\kappa_r} + O(e^{-\kappa_r t})$$

and on continuing the integrations

$$\sigma^{-s-1} e^{-\kappa_r t} = \sigma^{-s} \frac{1}{\kappa_r} = \frac{1}{\kappa_r} \frac{t^s}{s!}.$$

Thus $$x_r = \frac{\kappa_{r-1}}{\kappa_r} t^s = \frac{\kappa_{r-1}}{\kappa_r} x_{r-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Classifying elements into long-lived and short-lived according as $\kappa_r t$ is small or large for them, we find that the quantity of the first long-lived element after uranium is proportional to t, the next to t^2, and so on. All β-ray products are short-lived when t has ordinary values.

Radium is the third degeneration product of uranium. In rock specimens the time elapsed since formation is usually such that the relations (5) have become established. As a matter of observation the numbers of atoms of radium and uranium are found to be in the constant ratio $3{\cdot}58 \times 10^{-7}$. This determines κ/κ_s. Also the rate of break-up of radium is known directly: in fact

$$1/\kappa_s = 2280 \text{ years.}$$

Hence
$$1/\kappa = 6{\cdot}37 \times 10^9 \text{ years.}$$

This gives the rate of disintegration of uranium itself.

A number of specimens of uranium compounds were carefully freed from radium by Soddy, and then kept for ten years. It was found that new radium was formed; the amount found varied as the square of the time. This would suggest that of the two elements between uranium and radium in the series one was long-lived and the other short-lived. Actually, however, it is known independently that both are long-lived. The first, however, is chemically inseparable from ordinary uranium, and therefore was present in the original specimens; initially, instead of $x_1 = 0$, we have

$$x_1 = \frac{\kappa}{\kappa_1} u_0.$$

For the next element, ionium, we have

$$x_2 = \kappa_1 \sigma^{-1} x_1 = \kappa u_0 t,$$

the variation of x_1 being inappreciable in the time involved. Also

$$x_3 = \kappa_2 \sigma^{-1} x_2 = \tfrac{1}{2} \kappa \kappa_2 u_0 t^2.$$

Soddy* found that 3 kilograms of uranium in $10{\cdot}15$ years gave 202×10^{-12} gm. of radium. Hence, allowing for the difference of atomic weights,

$$x_3/u_0 = 7{\cdot}1 \times 10^{-14}$$

and
$$\kappa_2 = 8{\cdot}64 \times 10^{-6}/\text{year}; \quad 1/\kappa_2 = 1{\cdot}16 \times 10^5 \text{ years.}$$

This gives the rate of degeneration of ionium. Soddy gets a slightly lower value for $1/\kappa_2$ from more numerous data.

3.7. *Some dynamical applications.* Suppose we have a dynamical system specified by equations such as those in 1.72, save that x is replaced by t, and that the system is at first at rest and then disturbed by a force S_r applied to the coordinate y_r. The subsidiary equations are

$$\Sigma_s e_{ms} y_s = 0 \ (m \neq r); \quad \Sigma_s e_{ms} y_s = S_r \ (m = r). \quad \text{......(1)}$$

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* *Phil. Mag.* (6) 38, 1919, 483–488.

Writing Δ for the determinant formed by the e's, and E_{rs} for the minor of e_{rs} in this determinant, we have the operational solution

$$y_s = \frac{E_{rs}}{\Delta} S_r. \qquad \text{.........................(2)}$$

If the determinant Δ is symmetrical, so that

$$E_{rs} = E_{sr} \qquad \text{.........................(3)}$$

we see that a given force S_r applied to the coordinate y_r will produce precisely the same variation in y_s as the same force would produce in y_r if it was applied to y_s. Thus we have a reciprocity theorem applicable to all non-gyroscopic systems; linear damping does not invalidate the argument if the terms introduced by friction contribute only to the leading diagonal of Δ.

If the forces reduce to an impulse, so that S_r can be replaced by σJ_r, the solution becomes

$$y_s = \frac{E_{rs}}{\Delta} \sigma J_r. \qquad \text{.........................(4)}$$

We can evaluate the initial velocities by expanding in descending powers of σ. The first term is

$$y_s = \frac{A_{rs}}{A} \frac{1}{\sigma} J_r \qquad \text{.........................(5)}$$

where A is the determinant formed by the a's, and A_{rs} the minor of a_{rs} in it. Hence the initial velocities, found by operating on this with σ, are

$$\sigma y_s = \frac{A_{rs}}{A} J_r. \qquad \text{.........................(6)}$$

Now the constants a_{rs} are merely twice the coefficients in the kinetic energy, which is a quadratic form. Hence the determinant A is symmetrical whether the system is gyroscopic or not, and the reciprocity theorem for impulses and the velocities produced by them is proved.

The subsequent motion can be investigated by interpreting according to the partial-fraction rule. But let us consider the simple case where the system is non-gyroscopic and frictionless. Then

$$\Delta = A \Pi (\sigma^2 + \alpha^2), \qquad \text{.........................(7)}$$

where the α's are the speeds of the normal modes. Then

$$y_s = \Sigma_\alpha \frac{E_{rs}(-\alpha^2)}{\alpha \, (d\Delta/d\sigma^2)_{\sigma^2=\alpha^2}} J_r \sin \alpha t, \qquad \text{.....................(8)}$$

where $E_{rs}(-\alpha^2)$ denotes the result of putting $-\alpha^2$ for σ^2 in E_{rs}. The contribution of the α mode to the initial rate of change of y_s is therefore

$$v_{s\alpha} = \frac{E_{rs}(-\alpha^2)}{(d\Delta/d\sigma^2)_{\sigma^2=\alpha^2}} J_r. \quad \text{.........................(9)}$$

An immediate consequence of the presence of the factor $E_{rs}(-\alpha^2)$ in the numerator is that if an impulse is applied at a node of any normal mode, that mode will be absent from the motion generated.

Another illustration is provided by Lamb's discussion* of the waves generated in a semi-infinite homogeneous elastic solid by an internal disturbance. The normal modes of such a system include a type of waves known as Rayleigh waves. These may be of any length, and involve both compressional and distortional movement; if the depth is z, the amplitude of the compressional movement in a given wave is proportional to $e^{-\alpha z}$, and that of the distortional movement to $e^{-\beta z}$, where α and β depend only on the wave-length. Lamb found that if the original disturbance was an expansive one at a depth f, the amplitude of the motion at the surface contained a factor $e^{-\alpha f}$; but if the original disturbance was purely distortional, the corresponding amplitude contained a factor $e^{-\beta f}$. These factors are the same as would occur in the compression and distortion respectively at depth f in a Rayleigh wave with given amplitude at the surface.

CHAPTER IV

WAVE MOTION IN ONE DIMENSION

4.1. In a large class of physical problems we meet with the differential equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0, \quad \text{.........................(1)}$$

where t is the time, x the distance from a fixed point or a fixed plane, y the independent variable, and c a known velocity. Let us consider the solution of this equation first with regard to the transverse vibrations of a stretched string. In this case we know that

$$c^2 = P/m, \quad \text{...............................(2)}$$

where P is the tension and m the mass per unit length. Write σ for $\partial/\partial t$, and p for $\partial/\partial x$. Suppose that at time zero

$$y = f(x); \quad \frac{\partial y}{\partial t} = F(x), \quad \text{....................(3)}$$

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* *Phil. Trans.* A, 203, 1–42, 1904.

where f and F are known functions of x; that is, we are given the initial displacement and velocity of the string at all points of its length. Then we are led by our previous rules to consider the subsidiary equation

$$(\sigma^2 - c^2p^2)y = \sigma^2 f(x) + \sigma F(x) \quad \text{......(4)}$$

or

$$y = \frac{\sigma^2}{\sigma^2 - c^2p^2} f(x) + \frac{\sigma}{\sigma^2 - c^2p^2} F(x). \quad \text{......(5)}$$

But

$$\frac{\sigma^2}{\sigma^2 - c^2p^2} = \tfrac{1}{2}\left(\frac{\sigma}{\sigma - cp} + \frac{\sigma}{\sigma + cp}\right)$$
$$= \tfrac{1}{2}(e^{cpt} + e^{-cpt}), \quad \text{......(6)}$$

$$\frac{\sigma}{\sigma^2 - c^2p^2} = \frac{1}{2cp}\left(\frac{\sigma}{\sigma - cp} - \frac{\sigma}{\sigma + cp}\right)$$
$$= \frac{1}{2c}(e^{cpt} - e^{-cpt})\frac{1}{p}. \quad \text{......(7)}$$

In (7) the $1/p$ has been put last in consequence of our rule that operations involving negative powers of p must be carried out before those involving positive powers. Hence

$$\frac{\sigma^2}{\sigma^2 - c^2p^2} f(x) = \tfrac{1}{2}(e^{cpt} + e^{-cpt}) f(x)$$
$$= \tfrac{1}{2}\{f(x+ct) + f(x-ct)\}, \quad \text{......(8)}$$

$$\frac{\sigma}{\sigma^2 - c^2p^2} F(x) = \frac{1}{2c}(e^{cpt} - e^{-cpt}) \int_0^x F(x)\,dx$$
$$= \frac{1}{2c}\left[\int_0^{x+ct} F(x)\,dx - \int_0^{x-ct} F(x)\,dx\right]$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} F(x)\,dx \quad \text{......(9)}$$

and therefore

$$y = \tfrac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} F(x)\,dx. \quad \text{...(10)}$$

This is D'Alembert's well-known solution.

This cannot, however, be the complete solution. Equation (1) has been assumed to hold at all points of the string; but if any external forces act these must be included on the right of the equation of motion. In such problems the ends of the string are usually fixed, and reactions at the ends are required to maintain this state; in the complete equations of motion these reactions should appear. They are unknown functions of t, and therefore necessitate a change in the mode of solution.

But D'Alembert's solution can be adapted so as to fulfil all the conditions. We notice that the solution consists of two waves travelling in opposite directions with velocity c. If the initial disturbance is confined to a region within the string separated by finite intervals from both ends, it will take a finite time before D'Alembert's solution gives a displacement at either end; hence no force is required to maintain the boundary conditions, and the solution will hold accurately until one of the waves reaches one end.

Again, the initial disturbance is specified only for points within the length of the string, that is, for values of x between 0 and l, say; and by the last paragraph D'Alembert's solution would be complete if the length were infinite. If then we consider an infinite string stretching from $-\infty$ to $+\infty$, and disturbed initially so that $f(x)$ and $F(x)$ are both antisymmetrical about both $x=0$ and $x=l$, y will vanish ever after at these points, so that no force is required to keep them fixed, and D'Alembert's solution for such an infinite string will therefore be equivalent to that for the actual string from $x=0$ to $x=l$. Thus we have a formal rule for finding the position of the string at any subsequent time: consider an infinite string with the actual values of the initial velocity and displacement between $x=0$ and $x=l$, but with the displacement and velocity outside this stretch so specified as to be antisymmetrical with regard to both $x=0$ and $x=l$; then D'Alembert's solution for the infinite string will be correct for the actual string for the same values of x.

4.2. We may approach the problem in another way. Taking the same subsidiary equation

$$\frac{\partial^2 y}{\partial x^2}-\frac{\sigma^2}{c^2}y=-\frac{\sigma^2}{c^2}f(x), \quad \text{.....................(1)}$$

except that the effects of the initial velocity will not be considered at present, let us solve with regard to x as if σ were a constant. The justification of this procedure is given later in 4.6. The boundary conditions are that $y=0$ when $x=0$ and when $x=l$. Using the method of variation of parameters, we assume that the solution is

$$y=A\cosh\frac{\sigma x}{c}+B\sinh\frac{\sigma x}{c}, \quad \text{..................(2)}$$

where A and B are functions of x subject to

$$A'\cosh\frac{\sigma x}{c}+B'\sinh\frac{\sigma x}{c}=0. \quad \text{......................(3)}$$

Substituting in the equation (1) we find

$$A' \sinh \frac{\sigma x}{c} + B' \cosh \frac{\sigma x}{c} = -\frac{\sigma}{c} f(x). \quad \text{..............(4)}$$

Hence

$$A' = \frac{\sigma}{c} f(x) \sinh \frac{\sigma x}{c}\ ;\ B' = -\frac{\sigma}{c} f(x) \cosh \frac{\sigma x}{c}. \quad \text{..........(5)}$$

Now y must vanish for all time when $x=0$; hence $A(0)=0$. Thus

$$A = \int_0^x \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma \xi}{c}\, d\xi \text{........................(6)}$$

Also y must vanish when $x=l$. Hence

$$B(l) \sinh \frac{\sigma l}{c} = -A(l) \cosh \frac{\sigma l}{c} \quad \text{..................(7)}$$

and
$$B(l) = -\coth \frac{\sigma l}{c} \int_0^l \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma \xi}{c}\, d\xi, \text{...............(8)}$$

giving
$$B(x) = -\coth \frac{\sigma l}{c} \int_0^l \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma \xi}{c}\, d\xi + \int_x^l \frac{\sigma}{c} f(\xi) \cosh \frac{\sigma \xi}{c}\, d\xi. \quad \text{....................(9)}$$

In all we find

$$\begin{aligned} y &= \int_0^x \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma \xi}{c} \left(\cosh \frac{\sigma x}{c} - \coth \frac{\sigma l}{c} \sinh \frac{\sigma x}{c} \right) d\xi \\ &\quad + \int_x^l \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma x}{c} \left(\cosh \frac{\sigma \xi}{c} - \coth \frac{\sigma l}{c} \sinh \frac{\sigma \xi}{c} \right) d\xi \\ &= \int_0^x \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma \xi}{c} \frac{\sinh \sigma (l-x)/c}{\sinh \sigma l/c}\, d\xi \\ &\quad + \int_x^l \frac{\sigma}{c} f(\xi) \sinh \frac{\sigma x}{c} \frac{\sinh \sigma (l-\xi)/c}{\sinh \sigma l/c}\, d\xi. \text{.....................(10)} \end{aligned}$$

Before carrying out the integration with regard to ξ, we can interpret the integrand by the partial-fraction rule. Each integral vanishes when $\sigma = 0$, and therefore no constant term appears in the solution. But $\sinh \sigma l/c$ vanishes when

$$\frac{\sigma l}{c} = r\iota\pi, \quad \text{...............................(11)}$$

where r is any integer, positive or negative. Hence

$$\begin{aligned} \frac{\sigma}{c} \sinh \frac{\sigma \xi}{c} \frac{\sinh \sigma (l-x)/c}{\sinh \sigma l/c} &= \sum_{r=-\infty}^{r=\infty} \frac{\dfrac{r\iota\pi}{l} \sinh \dfrac{r\iota\pi\xi}{l} \sinh r\iota\pi \dfrac{l-x}{l}}{\dfrac{cr\iota\pi}{l} \cosh (r\iota\pi) \dfrac{l}{c}}\, e^{r\iota\pi ct/l} \\ &= \sum_{r=1}^{\infty} \frac{2}{l} \sin \frac{r\pi\xi}{l} \sin \frac{r\pi x}{l} \cos \frac{r\pi ct}{l}. \quad \text{.........(12)} \end{aligned}$$

By symmetry the corresponding factor in the second integral has the same value. Hence

$$y = \sum_{r=1}^{\infty} \frac{2}{l} \int_0^l f(\xi) \sin \frac{r\pi\xi}{l}\, d\xi \,.\, \sin \frac{r\pi x}{l} \cos \frac{r\pi ct}{l}. \qquad \text{......(13)}$$

This is the solution given by the method of normal coordinates. Putting t zero, we obtain the Fourier sine series

$$f(x) = \frac{2}{l} \int_0^l f(\xi) \sin \frac{r\pi\xi}{l} \sin \frac{r\pi x}{l}\, d\xi. \qquad \text{..............(14)}$$

The effect of an initial velocity can be obtained by a similar process.

But in practice the solution by trigonometrical series is not often the most convenient form. It usually converges slowly; but what is more serious is that its form suggests little about the nature of the actual motion beyond the fact that it is periodic in time $2l/c$. To find the actual form of the string at any instant it is necessary to find some way of summing the series, which may be rather difficult. A more convenient method is often the following[*]. We have seen that the interpretation of any operator valid for positive values of the argument is equivalent to an integral in the complex plane, the path of integration being on the positive side of the imaginary axis. Then a factor cosech $\sigma l/c$ in the operator can be written

$$\frac{1}{\sinh \sigma l/c} = \frac{2e^{-\sigma l/c}}{1 - e^{-2\sigma l/c}} = 2e^{-\sigma l/c}\left(1 + e^{-2\sigma l/c} + e^{-4\sigma l/c} + \ldots\right) \qquad \text{...(15)}$$

and when interpreted by Bromwich's rule this will give rise to a series in powers of $e^{-2\kappa l/c}$ in the integrand. But since the real part of κ is everywhere positive on the path, the series is absolutely convergent, and integration term by term is justified. Hence it is legitimate to expand the operator in this way and to interpret term by term. Then for $\xi < x$,

$$\frac{\sinh \sigma\xi/c \sinh \sigma(l-x)/c}{\sinh \sigma l/c} = \tfrac{1}{2}e^{-\sigma(x-\xi)/c}\left(1 - e^{-2\sigma\xi/c}\right)$$

$$\left(1 - e^{-2\sigma(l-x)/c}\right)\left(1 + e^{-2\sigma l/c} + e^{-4\sigma l/c} + \ldots\right); \qquad \text{...(16)}$$

and for $\xi > x$,

$$\frac{\sinh \sigma x/c \sinh \sigma(l-\xi)/c}{\sinh \sigma l/c} = \tfrac{1}{2}e^{-\sigma(\xi-x)/c}\left(1 - e^{-2\sigma x/c}\right)$$

$$\left(1 - e^{-2\sigma(l-\xi)/c}\right)\left(1 + e^{-2\sigma l/c} + e^{-4\sigma l/c} + \ldots\right). \qquad \text{...(17)}$$

On multiplying out either expression we obtain a series of negative exponentials of the forms $e^{-\sigma(a-\xi)/c}$, with $a > \xi$, or $e^{-\sigma(\xi-a)/c}$, with $\xi > a$.

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[*] Heaviside, *Electromagnetic Theory*, 2, 108–114.

Then $$\int_{\xi_1}^{\xi_2} \frac{\sigma}{c} f(\xi)\, e^{-\sigma(a-\xi)/c}\, d\xi = \int f(\xi)\, de^{-\sigma(a-\xi)/c}. \quad \text{......(18)}$$

But according to our rule $e^{-\sigma(a-\xi)/c}$ is zero when $t-(a-\xi)/c$ is negative and unity when this quantity is positive. Hence the integral (18) is zero unless $a-ct$ lies between ξ_1 and ξ_2, and then it is equal to $f(a-ct)$.

Similarly $$\int_{\xi_1}^{\xi_2} \frac{\sigma}{c} f(\xi)\, e^{-\sigma(\xi-a)/c}\, d\xi = -\int f(\xi)\, de^{-\sigma(\xi-a)/c} \quad \text{.........(19)}$$

and $$\begin{aligned} e^{-\sigma(\xi-a)/c} &= 0 \text{ if } \xi > ct+a \\ &= 1 \text{ if } \xi < ct+a, \end{aligned}$$

so that (19) is zero unless $ct+a$ lies between ξ_1 and ξ_2, and then is equal to $f(ct+a)$.

It follows that at any instant the effect of the initial displacement $f(\xi)$ at the point ξ is zero except at a special set of points where one or other of the quantities $a \pm ct$ is equal to ξ. Since a does not involve ξ, we see that this way of expressing the solution reduces it to a set of waves moving in each direction with velocity c. The first three factors in (16) give

$$\tfrac{1}{2}\{e^{-\sigma(x-\xi)/c} - e^{-\sigma(x+\xi)/c} - e^{-\sigma\{(x-\xi)+2(l-x)\}/c} + e^{-\sigma\{(x+\xi)+2(l-x)\}/c}\} \quad \text{......(20)}$$

for values of x greater than ξ. The first term gives $\frac{1}{2}f(\xi)$ at time $(x-\xi)/c$, the second $-\frac{1}{2}f(\xi)$ at time $(x+\xi)/c$, the third $-\frac{1}{2}f(\xi)$ at time $\{x-\xi+2(l-x)\}/c$ and the fourth $\frac{1}{2}f(\xi)$ at time $\{x+\xi+2(l-x)\}/c$. The first term represents the direct wave from ξ to x, the second the wave reflected at $x=0$, the third that reflected at $x=l$, and the fourth one reflected first at $x=0$ and then again at $x=l$. The term $e^{-2\sigma l/c}$ in the last factor of (16) will give four further and similar pulses, each later by $2l/c$ than the corresponding pulse just found. These are pulses that have travelled twice more along the string, having been reflected once more at each end. Similarly we can proceed to the interpretation of later terms as pulses that have undergone still more reflexions.

The part of the solution arising from values of ξ greater than x may be treated similarly. The interpretation in terms of waves is exactly the same.

4.3. As a particular case let us consider a string of length l, originally drawn aside a distance η at the point $x=b$, so that initially it lies in two straight pieces, and then released. Then

$$\left.\begin{aligned} y_0 &= \eta x/b & 0<x<b \\ y_0 &= \eta(l-x)/(l-b) & b<x<l \end{aligned}\right\} \quad \text{..................(1)}$$

and the subsidiary equation is

$$\frac{\partial^2 y}{\partial x^2} - \frac{\sigma^2}{c^2} y = -\frac{\sigma^2}{c^2} y_0. \quad \text{.......................}(2)$$

The solution that vanishes at the ends is

$$\left.\begin{aligned} y &= \eta \frac{x}{b} + A \sinh \frac{\sigma x}{c} \sinh \frac{\sigma}{c}(l-b) \qquad & 0 < x < b \\ y &= \eta \frac{l-x}{l-b} + A \sinh \frac{\sigma b}{c} \sinh \frac{\sigma}{c}(l-x) \qquad & b < x < l \end{aligned}\right\}, \quad \text{......}(3)$$

where the constants have been chosen so as to make y continuous at $x = b$. Also a discontinuity in $\partial y/\partial x$ at this point would imply an infinite acceleration, which cannot persist. Hence we add the condition that $\partial y/\partial x$ shall be continuous, which gives

$$\eta\left(\frac{1}{b} + \frac{1}{l-b}\right) + \frac{\sigma A}{c}\left\{\cosh\frac{\sigma b}{c}\sinh\frac{\sigma}{c}(l-b) + \sinh\frac{\sigma b}{c}\cosh\frac{\sigma}{c}(l-b)\right\} = 0 \quad \text{.........}(4)$$

and on simplifying

$$A = -\frac{c}{\sigma}\frac{l}{b(l-b)}\eta \operatorname{cosech}\frac{\sigma l}{c}. \quad \text{...................}(5)$$

Thus

$$\left.\begin{aligned} y &= \frac{\eta l}{b(l-b)}\left\{\frac{l-b}{l}x - \frac{c}{\sigma}\frac{\sinh \sigma x/c \sinh \sigma(l-b)/c}{\sinh \sigma l/c}\right\} \qquad & 0 < x < b \\ y &= \frac{\eta l}{b(l-b)}\left\{\frac{b}{l}(l-x) - \frac{c}{\sigma}\frac{\sinh \sigma b/c \sinh \sigma(l-x)/c}{\sinh \sigma l/c}\right\} \qquad & b < x < l \end{aligned}\right\}. \quad \text{...}(6)$$

Interpreting by the partial-fraction rule, we notice that the contribution from $\sigma = 0$ just cancels the term independent of σ, while the rest gives*

$$y = \frac{2\eta l^2}{b(l-b)}\sum_{r=1}^{\infty}\frac{1}{r^2\pi^2}\sin\frac{r\pi b}{l}\sin\frac{r\pi x}{l}\cos\frac{r\pi ct}{l} \qquad 0 < x < l. \quad (7)$$

Alternatively, we can interpret the solution for $0 < x < b$ in exponentials, for

$$\frac{\sinh \sigma x/c \sinh \sigma(l-b)/c}{\sinh \sigma l/c} = \tfrac{1}{2}e^{-\sigma(b-x)/c}\left(1 - e^{-2\sigma x/c}\right)$$

$$\left(1 - e^{-2\sigma(l-b)/c}\right)\left(1 + e^{-2\sigma l/c} + e^{-4\sigma l/c} + \ldots\right) \text{...}(8)$$

and

$$\frac{c}{\sigma} = ct \qquad t > 0. \quad \text{..............................}(9)$$

Then

$$e^{-\sigma(b-x)/c}\frac{c}{\sigma} = ct - (b-x), \quad \text{.......................}(10)$$

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* Rayleigh, *Theory of Sound*, 1, 1894, 185.

when this is positive, and otherwise is zero. Thus y retains its initial value $\eta x/l$ unaltered until time $(b-x)/c$, when the wave from the part where $x > b$ begins to arrive. After that we shall have

$$y = \frac{\eta l}{b(l-b)}\left\{\frac{l-b}{l}x - \tfrac{1}{2}(ct-b+x)\right\} = \frac{\eta l}{b(l-b)}\left\{\left(\frac{1}{2}-\frac{b}{l}\right)x - \tfrac{1}{2}(ct-b)\right\}. \quad \text{.........(11)}$$

This solution remains correct until $ct = b + x$, when the wave reflected at $x = 0$ begins to arrive. Thenceforward

$$\left(e^{-\sigma(b-x)/c} - e^{-\sigma(b+x)/c}\right)\frac{c}{\sigma} = \{ct-(b-x)\} - \{ct-(b+x)\} = 2x. \quad \text{...(12)}$$

Hence $$y = \frac{\eta l}{b(l-b)}\left(\frac{l-b}{l}x - x\right) = -\frac{\eta x}{l-b} \quad \text{...............(13)}$$

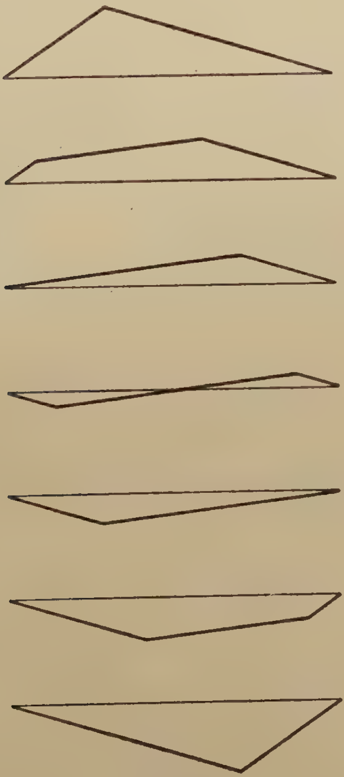

Fig. 6.

The part of the string reached by the first reflected wave is therefore parallel to the original position of the part where $b < x < l$.

When $ct-(b-x)+2(l-b)$, a wave reflected at $x=l$ arrives, and

$$y=\frac{\eta l}{b(l-b)}\left\{\left(\frac{1}{2}-\frac{b}{l}\right)x+\tfrac{1}{2}(ct+b)-l\right\}. \quad \text{.........(14)}$$

This holds until $ct=(b+x)+2(l-b)$; in the next phase

$$y=\frac{\eta l}{b(l-b)}x\left(1-\frac{b}{l}\right)=\frac{\eta x}{b}. \quad \text{...................(15)}$$

When $ct=2l$, the whole of the string is back in its original position; the term in $e^{-2\sigma l/c}$ then begins to affect the motion, and the whole process repeats itself. We see that at any instant the string is in three straight pieces. The two end pieces are parallel to the two portions of the string in its initial position, and are at rest. For the middle portion the gradient $\partial y/\partial x$ is the mean of those for the end portions, and the transverse velocity is $\pm\frac{1}{2}\frac{\eta lc}{b(l-b)}$. The middle portion is always either extending or withdrawing at each end with velocity c. [Fig. 6.]

4.4. A uniform heavy string of length $2l$ is fixed at the ends. A particle of mass m is attached to the middle of the string. Initially the string is straight. An impulse J is given to the particle. Find the subsequent motion of the particle. (Rayleigh, *Theory of Sound*, 1, 1894, 204.)

The differential equation of the motion of the string is

$$\frac{\partial^2 y}{\partial t^2}-c^2\frac{\partial^2 y}{\partial x^2}=0. \quad \text{..........................(1)}$$

Take x zero at the middle of the string. By symmetry we need consider only the range of values $0 < x < l$. Suppose that the displacement of the particle is η. When $t=0$, y and $\partial y/\partial t$ are zero except at $x=0$. The subsidiary equation therefore needs no additional terms. The solution is therefore

$$y=\eta\,\frac{\sinh\sigma(l-x)/c}{\sinh\sigma l/c}. \quad \text{.........................(2)}$$

If now P be the tension in the string, the equation of motion of the particle is

$$m\frac{\partial^2\eta}{\partial t^2}=2P\left(\frac{\partial y}{\partial x}\right)_{x=0}. \quad \text{.........................(3)}$$

If ρ be the mass of the string per unit length,

$$P=\rho c^2. \quad \text{...................................(4)}$$

On account of the initial conditions the subsidiary equation for the particle is

$$m\sigma^2\eta = 2P\left(\frac{\partial y}{\partial x}\right)_{x=0} + J\sigma$$

$$= -2P\eta\frac{\sigma}{c}\coth\frac{\sigma l}{c} + J\sigma. \qquad \text{.....................(5)}$$

Hence $$\eta = \frac{Jc}{m\sigma c + 2P\coth\sigma l/c}. \qquad \text{.....................(6)}$$

Put $$\frac{\rho l}{m} = k = \frac{\text{mass of half the string}}{\text{mass of the particle}}. \qquad \text{.....................(7)}$$

Then $$2P = kmc^2/l \qquad \text{.....................(8)}$$

and $$\eta = \frac{lJ/mc}{(\sigma l/c) + k\coth(\sigma l/c)}. \qquad \text{.....................(9)}$$

To interpret this solution by the partial-fraction rule, we recollect that the system is a stable one without dissipation, and therefore all the zeros of the denominator are purely imaginary. Putting

$$\sigma l/c = \iota\omega \qquad \text{.....................(10)}$$

we see that the zeros are the roots of

$$\omega = k\cot\omega. \qquad \text{.....................(11)}$$

There is a root between every two consecutive multiples of π, positive or negative; and the roots occur in pairs, each pair being numerically equal but opposite in sign. Then

$$\eta = \frac{lJ}{mc}\Sigma\frac{l}{\iota\omega c\,(1 + k\operatorname{cosec}^2\omega)\,(l/c)}e^{\iota\omega ct/l}, \qquad \text{............(12)}$$

the summation extending over all positive and negative values of ω,

$$= \frac{2lJ}{mc}\Sigma\frac{1}{\omega\,(1 + k\operatorname{cosec}^2\omega)}\sin(\omega ct/l), \qquad \text{............(13)}$$

the summation being now over positive values of ω.

If a root of (11) is $n\pi + \lambda$, where n is great and $\lambda < \pi$, we have

$$(n\pi + \lambda)\tan\lambda = k \qquad \text{.....................(14)}$$

and $$\lambda = k/n\pi \qquad \text{.....................(15)}$$

approximately. The series (13) converges like $\Sigma\frac{1}{\omega}\sin^2\omega$, or $\Sigma\frac{1}{n^3}$. It therefore converges fairly rapidly at the beginning, but more slowly later. Also the ω's are incommensurable, and therefore the motion does not repeat itself after a finite time; thus the labour of computation would be great.

To express the solution in terms of waves, it is convenient to change the unit of time to l/c, the time a wave takes to traverse half the length of the string. Also J/m may be replaced by V. Then

$$\eta = \frac{V}{\sigma + k \coth \sigma} = \frac{V(1-e^{-2\sigma})}{(\sigma+k)-(\sigma-k)e^{-2\sigma}}$$

$$= V\frac{1-e^{-2\sigma}}{\sigma+k}\left(1+\frac{\sigma-k}{\sigma+k}e^{-2\sigma}+\left(\frac{\sigma-k}{\sigma+k}\right)^2 e^{-4\sigma}+\dots\right)$$

$$= \frac{V}{\sigma+k}\left(1-\frac{2k}{\sigma+k}e^{-2\sigma}-\frac{2k(\sigma-k)}{(\sigma+k)^2}e^{-4\sigma}-\dots\right). \quad \text{...(16)}$$

The first term is zero for negative values of the time; for positive values it is equal to

$$\frac{V}{k}(1-e^{-kt}). \quad \text{.............................(17)}$$

After time 2 the second term no longer vanishes. We have

$$\frac{k}{(\sigma+k)^2} = \frac{1}{k} - \frac{\sigma}{k(\sigma+k)} - \frac{\sigma}{(\sigma+k)^2}$$

$$= \frac{1}{k} - \frac{1}{k}e^{-kt} - te^{-kt} \quad \text{.......................(18)}$$

and the second term is equivalent to

$$-\frac{2V}{k}\{1-e^{-k(t-2)}-k(t-2)e^{-k(t-2)}\} \text{ for } t>2. \quad \text{......(19)}$$

The third term is zero for $t<4$; for greater values it is easily found to be

$$\frac{2V}{k}[1-\{1+k(t-4)+k^2(t-4)^2\}e^{-k(t-4)}]. \quad \text{.........(20)}$$

The process may be extended to determine the motion up to any instant desired. The entry of a new term into the solution corresponds to the arrival of a new wave reflected at the ends.

4.5. A uniform heavy bar is hanging vertically from one end, and a mass m is suddenly attached at the lower end. Find how the tension at the upper end varies with the time. (Love, *Elasticity*, § 283.)

If x be the distance from the upper end, and y the longitudinal displacement, y satisfies the differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} - E\frac{\partial^2 y}{\partial x^2} = F, \quad \text{..........................(1)}$$

where ρ is the density, E Young's modulus, and F the external force per unit volume, in this case ρg. Put $E/\rho = c^2$, and let the displacement of a particle before the weight is attached be y_0. Then

$$-c^2 \frac{\partial^2 y_0}{\partial x^2} = g. \qquad \text{.............................(2)}$$

When $x = 0$, $y_0 = 0$; and if the bar be of length l, $\partial y_0/\partial x = 0$ when $x = l$.

Hence
$$y_0 = \frac{glx}{c^2}\left(1 - \tfrac{1}{2}\frac{x}{l}\right). \qquad \text{........................(3)}$$

After the weight is attached, we still have

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = g = -c^2 \frac{\partial^2 y_0}{\partial x^2}, \qquad \text{...................(4)}$$

while when $t = 0$, $y = y_0$ and $\partial y/\partial t = 0$. Hence the subsidiary equation is

$$\sigma^2 y - c^2 \frac{\partial^2 y}{\partial x^2} = \sigma^2 y_0 - c^2 \frac{\partial^2 y_0}{\partial x^2}, \qquad \text{...................(5)}$$

and the solution that vanishes with x is given by

$$y - y_0 = A \sinh \sigma x/c, \qquad \text{......................(6)}$$

where A is independent of x.

If ϖ be the cross section of the bar, the equation of motion of the mass m is

$$m \frac{\partial^2 y}{\partial t^2} = mg - E\varpi \frac{\partial y}{\partial x}, \qquad \text{......................(7)}$$

the derivatives being evaluated at $x = l$. The subsidiary equation is

$$m\sigma^2 y = mg + m\sigma^2 y_0 - E\varpi \frac{\partial y}{\partial x}, \qquad \text{.................(8)}$$

which gives on substituting for y from (6)

$$\left(\sigma^2 \sinh \frac{\sigma l}{c} + \frac{E\varpi\sigma}{mc} \cosh \frac{\sigma l}{c}\right) A = g. \qquad \text{..............(9)}$$

The tension at the upper end is

$$g\rho l + E\sigma A/c = g\rho l + \frac{Emg}{E\varpi \cosh(\sigma l/c) + mc\sigma \sinh(\sigma l/c)}. \qquad (10)$$

If k be the ratio of the mass of the weight to that of the bar,

$$k = \frac{m}{\rho\varpi l}; \quad \frac{mc}{E\varpi} = k\frac{l}{c}; \qquad \text{......................(11)}$$

and the tension is

$$\frac{gm}{\varpi}\left[\frac{1}{k}+\frac{1}{\cosh(\sigma l/c)+k(\sigma l/c)\sinh\sigma l/c}\right]. \quad \text{.........(12)}$$

We see that $gm/\varpi k$ is the tension due to the weight of the bar alone, and gm/ϖ is the statical tension due to the added load alone. To evaluate the actual tension, we expand the operator in (5) in powers of $e^{-\sigma l/c}$. Taking l/c for the new unit of time, we have

$$\begin{aligned}\frac{1}{k\sigma\sinh\sigma+\cosh\sigma}&=\frac{2e^{-\sigma}}{(k\sigma+1)-(k\sigma-1)e^{-2\sigma}}\\&=\frac{2e^{-\sigma}}{k\sigma+1}\left[1+\frac{k\sigma-1}{k\sigma+1}e^{-2\sigma}+\left(\frac{k\sigma-1}{k\sigma+1}\right)^2e^{-4\sigma}+\dots\right].\end{aligned} \quad \text{......(13)}$$

The first term vanishes up to time unity, and afterwards is equal to

$$2\left(1-e^{-(t-1)/k}\right). \quad \text{.............................(14)}$$

This increases steadily up to time 3, when the next term enters. Again,

$$\begin{aligned}\frac{k\sigma-1}{(k\sigma+1)^2}&=-1+\frac{k\sigma}{k\sigma+1}+\frac{2k\sigma}{(k\sigma+1)^2}\\&=-1+e^{-t/k}+2(t/k)e^{-t/k}\end{aligned} \quad \text{...............(15)}$$

and the first two terms, when $t>3$, are equivalent to

$$\begin{aligned}&2\left(1-e^{-(t-1)/k}\right)-2\left[1-e^{-(t-3)/k}-2\{(t-3)/k\}e^{-(t-3)/k}\right]\\&=2e^{-(t-3)/k}\left[1+(2/k)(t-3)-e^{-2/k}\right].\end{aligned} \quad \text{...................(16)}$$

This has a maximum when

$$1+e^{-2/k}=\frac{2}{k}(t-3). \quad \text{........................(17)}$$

Equation (17) has a root less than 5 if

$$4/k>1+e^{-2/k}, \quad \text{............................(18)}$$

which is an equality if $k=2{\cdot}7$. Thus for $k=1$ or 2 the maximum tension will occur before $t=5$. If $k=1$ the maximum tension is when $t=3{\cdot}568$, and is equal to $3{\cdot}266\ gm/\varpi$—$1{\cdot}633$ times the statical tension. If $k=2$ the corresponding results are $t=4{\cdot}368$, $2{\cdot}520\ gm/\varpi$, and $1{\cdot}680$ times the statical tension.

The third term enters at time 5, and afterwards is equal to

$$2\left[1-e^{-(t-5)/k}-2\{(t-5)/k\}^2e^{-(t-5)/k}\right].$$

If $k=4$, the maximum stress is when $t=6{\cdot}183$, and is equal to $2{\cdot}29\ gm/\varpi$. The statical tension is $1{\cdot}25\ gm/\varpi$, so that the ratio is $1{\cdot}83$.

Love proceeds by a method of continuation, but the present method is much more direct, and probably less troublesome in application.

4.6. A general proof that the results given by the operational method, when applied to the vibrations of continuous systems, are actually correct, has not yet been constructed. It would be necessary to show that the solution actually satisfies the differential equation and that it gives the correct initial values of the displacement and the velocity at all points. The proof that it satisfies the initial conditions would be difficult in the most general case. We have seen that in one of the simplest problems, that of the uniform string with the ends fixed, the verification that the solution is valid for the most general initial displacement is equivalent to Fourier's theorem. But for more specific problems the operational solution is equivalent to a single integral, and the direct verification that it satisfies the initial conditions is usually fairly easy. To show that it satisfies the differential equation, it would be natural to differentiate under the integral sign and substitute in the equation. But in practice it is usually found that the integrand, near the ends of the imaginary axis, is only small like some low power of $1/\kappa$ (the second in the problems of 4.3, 4.4, and 4.5), and consequently the integrals found by differentiating twice under the integral sign do not converge. But we can proceed as follows. If a part of our solution is

$$y = f(\sigma)\, e^{\sigma x/c} = \frac{1}{2\pi\iota}\int_L f(\kappa) \exp \kappa\left(t + \frac{x}{c}\right)\frac{d\kappa}{\kappa},$$

and this integral is intelligible for all values of x and t within a certain range, we have

$$\frac{\partial^2 y}{\partial t^2} = \operatorname*{Lim}_{h\to 0} \frac{y_{t+h} - 2y_t + y_{t-h}}{h^2}$$

$$= \operatorname*{Lim}_{h\to 0} \frac{1}{2\pi\iota}\int_L f(\kappa) \exp \kappa\left(t + \frac{x}{c}\right)\frac{e^{\kappa h} - 2 + e^{-\kappa h}}{h^2}\frac{d\kappa}{\kappa}.$$

Also
$$c^2 \frac{\partial^2 y}{\partial x^2} = c^2 \operatorname*{Lim}_{h\to 0} \frac{y_{x+ch} - 2y_x + y_{x-ch}}{c^2 h^2}$$

$$= \operatorname*{Lim}_{h\to 0} \frac{1}{2\pi\iota}\int_L f(\kappa) \exp \kappa\left(t + \frac{x}{c}\right)\frac{e^{\kappa h} - 2 + e^{-\kappa h}}{h^2}\frac{d\kappa}{\kappa}.$$

The two integrals are identical before we proceed to the limit, and therefore their limiting values are the same. Hence

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

and the differential equation is satisfied. This argument does not assume that the derivatives of y are expressible as convergent integrals, but only that they exist.

The argument breaks down at points where the second derivatives do not exist; as for instance in 4.3 at the points where the slope of the curve formed by the string suddenly changes. At these points there is a discontinuity in the transverse component of the tension, so that the point has momentarily an infinite acceleration. This is why the velocity changes discontinuously when a wave arrives. The momentum of a given stretch of the string, however, varies continuously; the difficulty is the fault of the representation by a differential equation, not of the method of solution.

CHAPTER V

CONDUCTION OF HEAT IN ONE DIMENSION

5.1. The rate of transmission of heat across a surface by conduction is equal to $-k\partial V/\partial n$ per unit area, where V is the temperature, k a constant of the material called the thermal conductivity, and dn an element of the normal to the surface. Hence we can show easily that in a uniform solid the rate of flow of heat into an element of volume $dx\,dy\,dz$ is

$$k\nabla^2 V.dx\,dy\,dz.$$

But the quantity of heat required to produce a rise of temperature dV in unit mass is $c\,dV$, where c is the specific heat, and therefore that required to produce a rise dV in unit volume is $\rho c\,dV$, where ρ is the density. Hence the equation of heat conduction is

$$\frac{\partial}{\partial t}(\rho c V) = k\nabla^2 V. \quad\text{.........................}(1)$$

If we put

$$k/\rho c = h^2; \quad\text{.........................}(2)$$

h^2 is called the thermometric conductivity, and the equation becomes

$$\frac{\partial V}{\partial t} = h^2\nabla^2 V. \quad\text{.........................}(3)$$

In addition, there may be some other source of heat. If this would suffice to raise the temperature by P degrees per second if it stayed where it was generated, a term P must be added to the right of (3).

It is usually convenient to write σ for $\partial/\partial t$, and

$$\sigma = h^2 q^2. \quad\text{.........................}(4)$$

The operational solutions are then functions of q; but q must be expressed again in terms of σ before interpreting.

5.2. Consider first a uniform rod, with its sides thermally insulated, and initially at temperature S. At time zero the end $x=0$ is cooled to temperature zero, and afterwards maintained at that temperature. The end $x=l$ is kept at temperature S. Find the variation of temperature at other points of the rod.

The problem being one-dimensional, the equation of heat conduction is

$$\frac{\partial V}{\partial t} - h^2 \frac{\partial^2 V}{\partial x^2} = 0, \qquad \text{.........................(1)}$$

while at time 0, V is equal to S. Hence the subsidiary equation is

$$\sigma V - h^2 \frac{\partial^2 V}{\partial x^2} = \sigma S, \qquad \text{.........................(2)}$$

or

$$\frac{\partial^2 V}{\partial x^2} - q^2 V = -q^2 S. \qquad \text{.........................(3)}$$

The end conditions are that

$$\left. \begin{aligned} V &= 0 \text{ when } x = 0, \\ V &= S \text{ when } x = l. \end{aligned} \right\} \qquad \text{.........................(4)}$$

Hence

$$V = S\left(1 - \frac{\sinh q(l-x)}{\sinh ql}\right). \qquad \text{.....................(5)}$$

The integrand is an even function of q, and therefore a single-valued function of σ. It has poles where

$$ql = \pm \iota n\pi; \text{ that is, } \sigma = -h^2 n^2 \pi^2 / l^2, \qquad \text{...............(6)}$$

where n is any integer. But the negative values of q give the same values of σ as the positive values, and therefore when we apply the partial-fraction rule we need consider only the positive values. The part arising from $\sigma = 0$ is

$$S\left(1 - \frac{l-x}{l}\right) = S\frac{x}{l}. \qquad \text{.........................(7)}$$

The general term is

$$-S \frac{\sinh \iota n\pi (l-x)/l}{(-h^2 n^2 \pi^2 / l^2) \cosh(\iota n\pi)\,(l^2/2h^2 \iota n\pi)} e^{-n^2\pi^2h^2t/l^2}$$

$$= S\frac{2}{n\pi} \sin\frac{n\pi x}{l} e^{-n^2\pi^2h^2t/l^2}, \qquad \text{......(8)}$$

and the complete solution is

$$V = S\left[\frac{x}{l} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi x}{l} e^{-n^2\pi^2h^2t/l^2}\right]. \qquad \text{...............(9)}$$

If $\pi h t^{\frac{1}{2}}/l$ is moderate, this series converges rapidly, and no more convenient solution could be desired. It evidently tends in the limit to the steady value Sx/l.

But if $\pi h t^{\frac{1}{2}}/l$ is small the convergence will be slow. In this case we may adopt a form of the expansion method applied to waves*. We can write (5) in the form

$$V = S\left[1 - e^{-qx}\left(1 - e^{-2q(l-x)}\right)\left(1 + e^{-2ql} + e^{-4ql} + \ldots\right)\right]. \quad \ldots\ldots(10)$$

For if we interpret this as an integral along the path L, the argument of q is between $\pm\frac{1}{4}\pi$ at all points of the path, and the series converges uniformly. Integration term by term is therefore justifiable, and we may interpret term by term. Now

$$qx = x\sigma^{\frac{1}{2}}/h, \quad \ldots\ldots(11)$$

and by 2.4 (16)

$$e^{-qx} = 1 - \operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}}. \quad \ldots\ldots(12)$$

Hence

$$V = S\left[\operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}} + \left(1 - \operatorname{Erf}\frac{2l-x}{2ht^{\frac{1}{2}}}\right) - \left(1 - \operatorname{Erf}\frac{2l+x}{2ht^{\frac{1}{2}}}\right) - \ldots\right] \ldots(13)$$

When w is great, $1 - \operatorname{Erf} w$ is small compared with e^{-w^2}. If then $x/2ht^{\frac{1}{2}}$ is moderate, but $l/2ht^{\frac{1}{2}}$ large, this series is rapidly convergent, and can in most cases be reduced to its first term. This solution is therefore convenient in those cases where (9) is not.

5.3. *One-dimensional flow of heat in a region infinite in both directions.* First suppose that at time 0 the distribution of temperature is given by

$$V = H(x). \quad \ldots\ldots(1)$$

We have just seen that the function expressed in operational form by

$$e^{-qx} = 1 - \operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}}, \quad \ldots\ldots(2)$$

satisfies the differential equation

$$\frac{\partial y}{\partial t} - h^2\frac{\partial^2 y}{\partial x^2} = 0, \quad \ldots\ldots(3)$$

for positive values of x; and by symmetry it will also satisfy it for negative values of x, since the function and its derivatives with regard to x are continuous when $x = 0$. Also when t tends to zero this function tends to zero for all positive values of x, and to 2 for all negative values. It follows from these facts that the function

$$\tfrac{1}{2}(2 - e^{-qx}) = \tfrac{1}{2}\left[1 + \operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}}\right] \quad \ldots\ldots(4)$$

satisfies the differential equation for all values of x and all positive values of t; and when t tends to zero the function tends to zero for negative values of x, and to unity for positive values. Hence this function is the solution required.

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* Heaviside, *Electromagnetic Theory*, 2, 69–79, 287–8.

Suppose now that the initial distribution of temperature is

$$V = f(x), \quad \text{.................................(5)}$$

where $f(x)$ is any known function. Then this is equivalent to

$$V = \int_{\xi=-\infty}^{\infty} f(\xi)\, d\{H(\xi - x)\}. \quad \text{.....................(6)}$$

But the solution for positive values of the time that reduces to $H(\xi - x)$ when $t = 0$ is

$$\tfrac{1}{2}\left[1 + \operatorname{Erf}\frac{\xi - x}{2ht^{\frac{1}{2}}}\right].$$

Hence by the principle of the superposibility of solutions the solution of the more general problem is

$$V = \int_{\xi=-\infty}^{\infty} \tfrac{1}{2} f(\xi)\, d\left[1 + \operatorname{Erf}\frac{\xi - x}{2ht^{\frac{1}{2}}}\right], \quad \text{.....................(7)}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(\xi - x)^2/4h^2t} \frac{d\xi}{2ht^{\frac{1}{2}}}. \quad \text{.....................(8)}$$

Put now

$$\xi = x + 2ht^{\frac{1}{2}}\lambda. \quad \text{...........................(9)}$$

Then

$$V = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2ht^{\frac{1}{2}}\lambda)\, e^{-\lambda^2}\, d\lambda. \quad \text{..............(10)}$$

This is the general solution obtained by Fourier.

5.4. If the temperature is kept constant at $x = 0$, but the initial temperature is $f(x)$ for positive values of x, we may proceed as follows. If we consider instead a system infinite in both directions, but with the initial temperature specified for negative values of x so that

$$f(-x) = -f(x), \quad \text{..........................(1)}$$

we see that the temperature at $x = 0$ will be zero at all later instants and the solution of this problem will fit the actual one. Hence

$$V = \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(\xi)\, e^{-(\xi - x)^2/4h^2t} \frac{d\xi}{2ht^{\frac{1}{2}}} - \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(\xi)\, e^{-(\xi + x)^2/4h^2t} \frac{d\xi}{2ht^{\frac{1}{2}}}, \quad \text{......(2)}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-x/2ht^{\frac{1}{2}}}^{\infty} f(x + 2ht^{\frac{1}{2}}\lambda)\, e^{-\lambda^2} d\lambda - \frac{1}{\sqrt{\pi}} \int_{x/2ht^{\frac{1}{2}}}^{\infty} f(-x + 2ht^{\frac{1}{2}}\lambda)\, e^{-\lambda^2}\, d\lambda.$$

$$\text{.........(3)}$$

If in particular the initial temperature is everywhere S, $f(\xi)=S$, and

$$V = S.\frac{1}{\sqrt{\pi}}\int_{-x/2ht^{\frac{1}{2}}}^{x/2ht^{\frac{1}{2}}} e^{-\lambda^2}\,d\lambda$$

$$= S\,\mathrm{Erf}\,\frac{x}{2ht^{\frac{1}{2}}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Thus the solution of 5.2 is regenerated. In Kelvin's solution of the problem of the cooling of the earth, 5.3 (10) was adapted to a semi-infinite region in this way.

The value of $\partial V/\partial x$ at the end $x=0$ is obtainable by differentiating the solution valid for a semi-infinite region. In Kelvin's problem

$$V = S(1-e^{-qx}),$$

and

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = Sq = \frac{S}{h(\pi t)^{\frac{1}{2}}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The same result is found by differentiating (4).

We notice the curious fact that although the original exact solution for a finite region in 5.2 (5) is a single-valued function of σ, a square root of t appears in the approximate solution for a greatly extended region. The reason can be seen by differentiating the exact solution. It gives

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = Sq\coth ql, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

which is a single-valued function of σ. But when we use Bromwich's interpretation we find that

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = \frac{1}{2\pi\iota}S\int_L \frac{\kappa^{-\frac{1}{2}}}{h}\coth\left(\frac{\kappa^{\frac{1}{2}}l}{h}\right)e^{\kappa t}\,d\kappa, \quad \ldots\ldots\ldots\ldots(7)$$

which is again single valued; but if $\kappa^{\frac{1}{2}}$ is specified to be real and positive when κ is real and positive, it has a positive real part at all points on L, and therefore if l is great $\coth \kappa^{\frac{1}{2}}l/h$ is practically unity. The integral is therefore equivalent to

$$\frac{1}{2\pi\iota}S\int_L \frac{\kappa^{-\frac{1}{2}}}{h}e^{\kappa t}\,d\kappa, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

which is our interpretation of $\sigma^{\frac{1}{2}}S/h$, and is equal to $S/h\sqrt{(\pi t)}$. We could have started by specifying the sign of $\kappa^{\frac{1}{2}}$ to be negative when κ is positive, but then $\kappa^{\frac{1}{2}}$ and $\coth \kappa^{\frac{1}{2}}l/h$ would both have been simply reversed in sign, and the same answer would have been obtained.

5.5. *Imperfect cooling at the free end of a one-dimensional region.* With the initial conditions of 5.2, let us suppose that the end $x=l$ is maintained at temperature S as before, but that the end $x=0$ is not effectively cooled to temperature zero. Instead we suppose that it radiates away heat at a rate proportional to its temperature. At the same time heat is conducted to the end at a rate $k\partial V/\partial x$ per unit area. These effects must balance if the temperature at the surface is to vary continuously, so that instead of having $V=0$ at the end as before we shall have a relation of the form

$$\frac{\partial V}{\partial x}-\alpha V=0 \text{ at } x=0. \quad\text{..........................(1)}$$

The equation 5.2 (3) is unaltered, and the operational solution is

$$V=S\{1-A\sinh q(l-x)\} \quad\text{......................(2)}$$

where A is to be determined to satisfy (1). Hence

$$qA\cosh ql-\alpha(1-A\sinh ql)=0 \quad\text{...................(3)}$$

and

$$V=S\left\{1-\frac{\alpha\sinh q(l-x)}{q\cosh ql+\alpha\sinh ql}\right\}. \quad\text{..................(4)}$$

The roots in σ are real and negative, and we can proceed to an interpretation by the partial-fraction rule as usual. Or, using the expansion in "waves," we have

$$V=S\left[1-\frac{\alpha e^{-qx}}{q+\alpha}\left(1-e^{-2q(l-x)}\right)\left(1-\frac{q-\alpha}{q+\alpha}e^{-2ql}+\dots\right)\right]. \quad\text{...(5)}$$

If the length is great enough to make the terms involving e^{-2ql} inappreciable, we can reduce this to its first two terms, thus:

$$V=S\left[1-\frac{\alpha e^{-qx}}{q+\alpha}\right] \quad\text{.........................(6)}$$

If α is great, the solutions reduce to those of 5.2; this is to be expected, for (1) then implies that $V=0$ when $x=0$, which is the boundary condition adopted in 5.2. If α is small, V reduces to S; the reason is that this implies that there is no loss of heat from the end, and therefore the temperature does not change anywhere. For intermediate values we may proceed as follows. If

$$y=\frac{\alpha e^{-qx}}{q+\alpha} \quad\text{.................................(7)}$$

Bromwich's rule gives

$$y=\frac{1}{2\pi\iota}\int_L\frac{\alpha h}{\kappa^{\frac{1}{2}}+\alpha h}\exp\left(\kappa t-\frac{\kappa^{\frac{1}{2}}x}{h}\right)\frac{d\kappa}{\kappa}. \quad\text{...............(8)}$$

Put

$$\kappa=\lambda^2. \quad\text{..................................(9)}$$

The path of integration for λ is a curve going from $Re^{-\frac{1}{2}\pi\iota}$ to $Re^{\frac{1}{2}\pi\iota}$, where R is great, passing on the way within a finite distance of the origin on the positive side. Denote this path by M. Then

$$y = \frac{1}{\pi\iota}\int_M \frac{ah}{(\lambda+ah)\lambda}\exp\left(\lambda^2 t - \frac{\lambda x}{h}\right)d\lambda, \quad \text{..............}(10)$$

$$= \frac{1}{\pi\iota}\int_M \left(\frac{1}{\lambda} - \frac{1}{\lambda+ah}\right)\exp\left(\lambda^2 t - \frac{\lambda x}{h}\right)d\lambda. \quad \text{........}(11)$$

But
$$\frac{1}{\pi\iota}\int_M \frac{1}{\lambda}\exp\left(\lambda^2 t - \frac{\lambda x}{h}\right)d\lambda = \frac{1}{2\pi\iota}\int_L \exp\left(\kappa t - \frac{\kappa^{\frac{1}{2}}x}{h}\right)\frac{d\kappa}{\kappa}$$
$$= e^{-qx}$$
$$= 1 - \operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}}. \quad \text{...............}(12)$$

The second part of (11) can be written as an integral with regard to μ, where

$$\mu = \lambda + ah \quad \text{............................}(13)$$

along a path obtained by displacing M through a distance ah; but the integrand is regular between these paths and the route M may still be used. The second part is therefore

$$-\frac{1}{\pi\iota}\int_M \frac{1}{\mu}\exp\left\{\mu^2 t - \mu\left(2aht + \frac{x}{h}\right)\right\}\exp\left(a^2h^2t + ax\right)d\mu$$

$$= -\exp\left(a^2h^2t + ax\right)\left\{1 - \operatorname{Erf}\frac{x + 2ah^2t}{2ht^{\frac{1}{2}}}\right\} \quad \text{...........}(14)$$

and
$$y = 1 - \operatorname{Erf}\xi - \exp\left(\gamma^2 + ax\right)\{1 - \operatorname{Erf}(\xi + \gamma)\} \text{.........}(15)$$

where
$$\xi = x/2ht^{\frac{1}{2}};\ \gamma = aht^{\frac{1}{2}}. \quad \text{.........................}(16)$$

Hence
$$V = S\left[\operatorname{Erf}\xi + \exp\left(\gamma^2 + ax\right)\{1 - \operatorname{Erf}(\xi + \gamma)\}\right]. \quad \text{......}(17)$$

This is the same as Riemann's solution*.

The temperature at the end is

$$V_{x=0} = S\exp\gamma^2\{1 - \operatorname{Erf}\gamma\} \text{.....................}(18)$$

whence the temperature gradient at the end follows by (1). For small values of the time the temperature at the free end falls continuously; the temperature gradient there is not instantaneously infinite as in 5.4 (5). For great values of the time we can use the approximation 2.4 (21), giving

$$V_{x=0} = \frac{S}{ah\sqrt{(\pi t)}}\left[1 - \frac{1}{2}\left(\frac{1}{a^2h^2t}\right) + \frac{1}{2}\cdot\frac{3}{2}\cdot\left(\frac{1}{a^2h^2t}\right)^2 + \dots\right]. \quad \text{...}(19)$$

---

* Riemann-Weber, *Partielle Differentialgleichungen*, 2, 95–98, 1912.

This is equivalent to one found by Heaviside*. Heaviside gives also an expansion in a convergent series, suitable for small values of t, but it is probably less convenient than the equivalent expression (18) in finite terms. We see from (19) that the longer the time taken the better is the approximation to $(\partial V/\partial x)_{x=0}$ given by the simple theory of 5.2.

5.6. A long rod is fastened at the end $x=0$, the other end not being in contact with a conductor. Initially it is at temperature 0, but at time 0 the clamped end is raised to temperature S and kept there. Each part of the rod loses heat by radiation at a rate proportional to its temperature.

The differential equation is now

$$\frac{\partial V}{\partial t} = h^2\frac{\partial^2 V}{\partial x^2} - \alpha^2 V \ldots\ldots\ldots\ldots(1)$$

where α is a constant. Let us put

$$\sigma + \alpha^2 = h^2 r^2 \ldots\ldots\ldots\ldots(2)$$

and write the equation

$$\frac{\partial^2 V}{\partial x^2} - r^2 V = 0. \ldots\ldots\ldots\ldots(3)$$

At $x=l$ there is no conduction out of the rod, and therefore $\partial V/\partial x$ vanishes. Also $V=S$ when $x=0$. The solution is

$$V = S\frac{\cosh r(l-x)}{\cosh rl}. \ldots\ldots\ldots\ldots(4)$$

This can be expanded in powers of e^{-2rl}. The first term, which is the only one we require, is

$$V = Se^{-rx} \ldots\ldots\ldots\ldots(5)$$

$$= \frac{S}{2\pi\iota}\int_L \exp\left\{\kappa t - \frac{(\kappa+\alpha^2)^{\frac{1}{2}}x}{h}\right\}\frac{d\kappa}{\kappa}. \ldots\ldots\ldots\ldots(6)$$

Put

$$\kappa + \alpha^2 = \lambda^2. \ldots\ldots\ldots\ldots(7)$$

Then

$$V = \frac{S}{\pi\iota}\int_M \exp\left\{(\lambda^2-\alpha^2)t - \frac{\lambda x}{h}\right\}\frac{\lambda d\lambda}{\lambda^2-\alpha^2}$$

$$= \frac{S}{2\pi\iota}\int_M \exp\left\{(\lambda^2-\alpha^2)t - \frac{\lambda x}{h}\right\}\left\{\frac{1}{\lambda-\alpha}+\frac{1}{\lambda+\alpha}\right\}d\lambda. \ldots\ldots(8)$$

But if

$$\lambda = \alpha + \mu \ldots\ldots\ldots\ldots(9)$$

the term in $1/(\lambda-\alpha)$ becomes

$$\frac{S}{2\pi\iota}\int_M \exp\left\{\mu^2 t - \mu\left(\frac{x}{h} - 2\alpha t\right)\right\}\exp\left(-\frac{\alpha x}{h}\right)\frac{d\mu}{\mu}$$

$$= \tfrac{1}{2}S\exp\left(-\frac{\alpha x}{h}\right)\left\{1 - \operatorname{Erf}\frac{x-2\alpha ht}{2ht^{\frac{1}{2}}}\right\} \ldots\ldots\ldots\ldots(10)$$

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* *Electromagnetic Theory*, 2, 15.

as in 5.5 (12). The complete solution is

$$V=\tfrac{1}{2}S\left[\exp\left(-\frac{ax}{h}\right)\left\{1-\operatorname{Erf}\frac{x-2aht}{2ht^{\frac{1}{2}}}\right\}\right.$$
$$\left.+\exp\left(\frac{ax}{h}\right)\left\{1-\operatorname{Erf}\frac{x+2aht}{2ht^{\frac{1}{2}}}\right\}\right]\ldots\ldots\text{(11)}$$

When $a=0$ this reduces to

$$V=S\left(1-\operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\text{(12)}$$

so that the disturbance of temperature spreads along the rod, the time needed to produce a given rise of temperature at distance x from the end being proportional to x^2. If t is small enough to make $at^{\frac{1}{2}}$ small, the error functions will be practically unity except where x is not great compared with $2ht^{\frac{1}{2}}$. For such values of x the exponentials are nearly unity. Thus at first the heating proceeds almost as in the absence of radiation from the sides of the rod.

But if $at^{\frac{1}{2}}$ is great and $x/2ht^{\frac{1}{2}}$ small or moderate, the first error function in (11) is practically -1, and the second $+1$. Thus in these conditions

$$V=Se^{-ax/h}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(13)}$$

This is seen by a return to the original equation to be the solution corresponding to a steady state. This will hold so long as $at^{\frac{1}{2}}-x/2ht^{\frac{1}{2}}$ is large and positive, even if $x/2ht^{\frac{1}{2}}$ is itself large. The region where the approximation (13) is valid therefore spreads along the rod with velocity $2ah$; if $at^{\frac{1}{2}}$ has once become large, each point on the rod reaches a nearly steady temperature at a time rather greater than $x/2ah$.

If $at^{\frac{1}{2}}$ is large and $x/2ht$ still greater, the solution is nearly

$$V=\tfrac{1}{2}S\,e^{-ax/h}\left\{1-\operatorname{Erf}\frac{x-2aht}{2ht^{\frac{1}{2}}}\right\}\ldots\ldots\ldots\ldots\text{(14)}$$

If further the argument of the error function is large, we can write

$$V=\frac{1}{2\sqrt{\pi}}S\,e^{-ax/h}\exp\left\{-\frac{(x-2aht)^2}{4h^2t}\right\}\frac{2ht^{\frac{1}{2}}}{x-2aht}$$
$$=\frac{1}{\sqrt{\pi}}S\exp\left\{-\frac{x^2}{4h^2t}-a^2t\right\}\frac{ht^{\frac{1}{2}}}{x-2aht}.\ \ldots\ldots\ldots\ldots\ldots\text{(15)}$$

In the regions that have not reached their steady state, the temperature resembles that for the problem without radiation, except that the small factor $\frac{1}{2} e^{-a^2 t} \frac{x}{x-2aht}$ must be introduced.

5.7. *The cooling of the earth.* Cooling in the earth since it first became solid has probably not had time to become appreciable except within a layer whose thickness is small compared with the radius. It is therefore legitimate to neglect the effects of curvature and treat the problem as one-dimensional. Radiation from the outer surface must have soon reduced the temperature there to that maintained by solar radiation, so that we may suppose the surface temperature to be constant and adopt it for our zero of temperature. The chief difference from the problem of 5.2 is that we must allow for the heating effect of radioactivity in the outer layers. Suppose first that the quantity P defined in 5.1 is equal to a constant A down to a depth H, and zero below that depth. Take the initial temperature to be $S+mx$, where m is a constant. Then the subsidiary equation is

$$\left.\begin{aligned} \frac{\partial^2 V}{\partial x^2} - q^2 V &= -\frac{A}{h^2} - q^2 (S+mx) \qquad && 0<x<H \\ \text{and} \qquad &= -q^2 (S+mx) && H<x, \end{aligned}\right\} \quad \ldots\ldots\ldots(1)$$

and the solutions are

$$\left.\begin{aligned} V &= \frac{A}{q^2 h^2} + S + mx + Be^{-qx} + Ce^{qx} \qquad && 0<x<H \\ \text{and} \qquad &= S + mx + De^{-qx} && H<x. \end{aligned}\right\} \quad \ldots\ldots\ldots(2)$$

A term in e^{qx} is not required in the solution for great depths, because it would imply that the temperature there suddenly dropped a finite amount at all depths, however great, in consequence of a disturbance near the surface. The conditions to determine B, C, and D are that V vanishes at $x=0$, and that V and $\partial V/\partial x$ are continuous at $x=H$. Hence

$$B + C + S + A/\sigma = 0, \quad \ldots\ldots\ldots(3)$$

$$Be^{-qH} + Ce^{qH} + A/\sigma = De^{-qH}, \quad \ldots\ldots\ldots(4)$$

$$Be^{-qH} - Ce^{qH} = De^{-qH}. \quad \ldots\ldots\ldots(5)$$

Solving and substituting in (2), we find

$$\left.\begin{aligned} V &= S(1-e^{-qx}) + mx + \frac{A}{\sigma}\{1 - e^{-qx} - e^{-qH}\sinh qx\} \qquad && 0<x<H \\ V &= S(1-e^{-qx}) + mx + \frac{A}{\sigma}(\cosh qH - 1)\, e^{-qx} && x>H. \end{aligned}\right\} \quad (6)$$

These solutions involve terms of the form $\frac{1}{\sigma}e^{-qa}$, where a is positive. To interpret, we write

$$\frac{1}{\sigma}e^{-qa} = \frac{1}{2\pi\iota}\int_L \exp\left(\kappa t - \frac{\kappa^{\frac{1}{2}}a}{h}\right)\frac{d\kappa}{\kappa^2}$$

$$= \frac{1}{\pi\iota}\int_M \exp\left(\lambda^2 t - \frac{\lambda a}{h}\right)\frac{d\lambda}{\lambda^3}$$

$$= \frac{1}{2\pi\iota}\int_M \left(\frac{2t}{\lambda} - \frac{a}{h\lambda^2}\right)\exp\left(\lambda^2 t - \frac{\lambda a}{h}\right)d\lambda \quad \text{.........(7)}$$

on integrating by parts. But

$$\frac{1}{\pi\iota}\int_M \exp\left(\lambda^2 t - \frac{\lambda a}{h}\right)\frac{d\lambda}{\lambda} = e^{-qa} \quad \text{..................(8)}$$

and a further integration by parts gives

$$\frac{1}{2\pi\iota}\int_M \exp\left(\lambda^2 t - \frac{\lambda a}{h}\right)\frac{d\lambda}{\lambda^2} = \frac{1}{2\pi\iota}\int_M \left(2t - \frac{a}{h\lambda}\right)\exp\left(\lambda^2 t - \frac{\lambda a}{h}\right)d\lambda$$

$$= htqe^{-qa} - \frac{a}{2h}e^{-qa}. \quad \text{...(9)}$$

In all
$$\frac{1}{\sigma}e^{-qa} = \left(t + \frac{a^2}{2h^2}\right)\left(1 - \mathrm{Erf}\frac{a}{2ht^{\frac{1}{2}}}\right) - \frac{a}{h}\sqrt{\frac{t}{\pi}}\,e^{-a^2/4h^2t}. \quad \text{...(10)}$$

Thus we can obtain a solution in finite terms, which is easily seen to be identical with that previously obtained by a more laborious method*.

The temperature gradient at the surface is found most easily from (6); we have

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = Sq + m + \frac{A}{\sigma}(q - qe^{-qH}). \quad \text{................(11)}$$

But
$$\frac{1}{q}e^{-qH} = \frac{h}{2\pi\iota}\int_L \exp\left(\kappa t - \frac{\kappa^{\frac{1}{2}}H}{h}\right)\frac{d\kappa}{\kappa^{\frac{3}{2}}} \quad \text{.....................(12)}$$

$$= \frac{h}{\pi\iota}\int_M \exp\left(\lambda^2 t - \frac{\lambda H}{h}\right)\frac{d\lambda}{\lambda^2}$$

$$= 2h^2tqe^{-qH} - He^{-qH}$$

$$= 2h(t/\pi)^{\frac{1}{2}}e^{-H^2/4h^2t} - H\left(1 - \mathrm{Erf}\frac{H}{2ht^{\frac{1}{2}}}\right) \quad \text{......(13)}$$

and therefore

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = \frac{S}{h(\pi t)^{\frac{1}{2}}} + m + \frac{A}{h^2}\left[H - H\,\mathrm{Erf}\frac{H}{2ht^{\frac{1}{2}}} + 2h\left(\frac{t}{\pi}\right)^{\frac{1}{2}}\left(1 - e^{-H^2/4h^2t}\right)\right]. \quad \text{.........(14)}$$

This is identical with the result given in earlier works of mine except that in these the factor 2 was omitted from the last term.

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* Jeffreys, *Phil. Mag.* 32, 575–591, 1916; *The Earth*, 84.

In the actual problem a considerable simplification arises from the fact that H is small compared with $2ht^{\frac{1}{2}}$. On this account we can expand the solutions in powers of H, and retain only the earlier terms. Thus for the surface temperature gradient we have

$$\left(\frac{\partial V}{\partial x}\right)_{x=0} = Sq + m + \frac{A}{h^2 q^2}\cdot q\cdot\left(qH - \tfrac{1}{2}q^2H^2\right)$$

$$= \frac{S}{h\,(\pi t)^{\frac{1}{2}}} + m + \frac{AH}{h^2}\left(1 - \frac{H}{2h\,(\pi t)^{\frac{1}{2}}}\right) \quad\text{.........(15)}$$

and for the temperature at depths greater than H

$$V = S\left(1 - e^{-qx}\right) + mx + \frac{A}{h^2q^2}\cdot\tfrac{1}{2}q^2H^2 \,.\, e^{-qx}$$

$$= mx + \left(S - \frac{AH^2}{2h^2}\right)\operatorname{Erf}\frac{x}{2ht^{\frac{1}{2}}} + \frac{AH^2}{2h^2}. \quad\text{............(16)}$$

5·71. An alternative possibility is that the radioactive generation of heat, instead of being confined entirely to a uniform surface layer, may decrease exponentially with depth. In this case the subsidiary equation is

$$\frac{\partial^2 V}{\partial x^2} - q^2 V = -\frac{A}{h^2}e^{-\alpha x} - q^2\left(S + mx\right) \quad\text{..............(1)}$$

at all depths. We already know the part of the solution contributed by $S + mx$. The remainder is

$$W = \frac{A}{h^2}\,\frac{e^{-\alpha x} - e^{-qx}}{q^2 - \alpha^2} \quad\text{..........................(2)}$$

But
$$\frac{e^{-\alpha x}}{h^2\left(q^2 - \alpha^2\right)} = \frac{e^{-\alpha x}}{\sigma - h^2\alpha^2} = \frac{1}{h^2\alpha^2}e^{-\alpha x}\left(e^{h^2\alpha^2 t} - 1\right) \quad\text{...........(3)}$$

and
$$-\frac{\alpha^2 e^{-qx}}{q^2 - \alpha^2} = \tfrac{1}{2}\left(\frac{\alpha}{q + \alpha} - \frac{\alpha}{q - \alpha}\right)e^{-qx}$$

$$= \tfrac{1}{2}\left[e^{-qx} - \exp\left(\gamma^2 + \alpha x\right)\left\{1 - \operatorname{Erf}\left(\lambda + \gamma\right)\right\}\right]$$

$$+ \tfrac{1}{2}\left[e^{-qx} - \exp\left(\gamma^2 - \alpha x\right)\left\{1 - \operatorname{Erf}\left(\lambda - \gamma\right)\right\}\right] \quad\text{........ (4)}$$

where
$$\lambda = x/2ht^{\frac{1}{2}};\ \gamma = \alpha h t^{\frac{1}{2}}. \quad\text{.........................(5)}$$

Hence
$$W = \frac{A}{h^2\alpha^2}\left\{\exp\left(\gamma^2 - \alpha x\right) - \exp\left(-\alpha x\right)\right\}$$

$$+ \frac{A}{h^2\alpha^2}\left[\left(1 - \operatorname{Erf}\lambda\right) - \tfrac{1}{2}\exp\left(\gamma^2 + \alpha x\right)\left\{1 - \operatorname{Erf}\left(\lambda + \gamma\right)\right\}\right.$$

$$\left. - \tfrac{1}{2}\exp\left(\gamma^2 - \alpha x\right)\left\{1 - \operatorname{Erf}\left(\lambda - \gamma\right)\right\}\right]$$

$$= \frac{A}{h^2\alpha^2}\left[1 - \operatorname{Erf}\lambda - e^{-\alpha x} - \tfrac{1}{2}\exp\left(\gamma^2 + \alpha x\right)\left\{1 - \operatorname{Erf}\left(\lambda + \gamma\right)\right\}\right.$$

$$\left. + \tfrac{1}{2}\exp\left(\gamma^2 - \alpha x\right)\left\{1 - \operatorname{Erf}\left(\gamma - \lambda\right)\right\}\right] \quad (6)$$

which is the same as the solution obtained by Ingersoll and Zobel*.

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* *Mathematical Theory of Heat Conduction*, Ginn, 1913.

The contribution of radioactivity to the temperature gradient at the surface is

$$\left(\frac{\partial W}{\partial x}\right)_{x=0} = \frac{A}{h^2}\,\frac{1}{q+\alpha} \qquad \dots\dots\dots\dots(7)$$

$$= \frac{A}{h^2 \alpha}\{1-\exp\gamma^2(1-\mathrm{Erf}\,\gamma)\}. \qquad \dots\dots\dots\dots(8)$$

When γ is great, as it actually is, we have

$$\left(\frac{\partial W}{\partial x}\right)_{x=0} = \frac{A}{h^2 \alpha}\left(1-\frac{1}{\gamma\sqrt{\pi}}\right) = \frac{A}{h^2 \alpha}\left(1-\frac{1}{\alpha h\sqrt{(\pi t)}}\right) \dots\dots(9)$$

5.8. The justification of the method is easier in problems of heat conduction than in those of the last chapter, because the integrands always contain a factor $\exp\left(-\frac{\kappa^{\frac{1}{2}}x}{h}\right)$. This tends to zero when κ tends to $c \pm \iota\infty$ in such a way that the integrals obtained by differentiating under the integral sign always converge, and can therefore be substituted in the differential equation directly. But the integrals for the temperature are of the form

$$V = \frac{1}{2\pi\iota}\int f(\kappa)\exp\left(\kappa t - \frac{\kappa^{\frac{1}{2}}x}{h}\right)\frac{d\kappa}{\kappa},$$

and when we substitute in the equation

$$\frac{\partial V}{\partial t} - h^2\frac{\partial^2 V}{\partial x^2} = 0$$

the integrand vanishes identically.

CHAPTER VI

PROBLEMS WITH SPHERICAL OR CYLINDRICAL SYMMETRY

6.1. So far we have treated only problems of wave transmission or conduction of heat in one dimension. If our system has spherical symmetry, the equation of transmission of sound takes the form

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2\nabla^2\Phi = 0 \qquad \dots\dots\dots\dots(1)$$

where Φ is the velocity potential, and

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi). \qquad \dots\dots\dots\dots(2)$$

The differential equation is therefore equivalent to

$$\frac{\partial^2}{c^2\partial t^2}(r\Phi)-\frac{\partial^2}{\partial r^2}(r\Phi)=0. \quad\text{......................(3)}$$

This is of the same form as the equation of transmission of sound in one dimension, $r\Phi$ taking the place of Φ; but differences of treatment arise from differences in the boundary conditions. A similar transformation can of course be applied to the equation of heat conduction.

If the symmetry is cylindrical, we take Cartesian coordinates x, y, z, and put

$$x^2+y^2=\varpi^2,\quad y=x\tan\phi, \quad\text{......................(4)}$$

and if Φ is independent of z and ϕ we have

$$\nabla^2\Phi=\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\Phi}{\partial\varpi}\right), \quad\text{........................(5)}$$

which is capable of no simple transformation analogous to that just given for the case of spherical symmetry. This fact gives rise to striking differences between the phenomena of wave motion in two and three dimensions.

6.2. Consider first a spherical region of high pressure, surrounded by an infinitely extended region of uniform pressure; the boundary between them is solid, and the whole is at rest. Suddenly the boundary is annihilated. Find the subsequent motion. The problem is that of an explosion wave. We suppose the motion small enough to permit the neglect of squares of the displacements. At all points

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial r^2}\right)r\Phi=0. \quad\text{........................(1)}$$

Initially there is no motion, so that Φ is constant, and may be taken as zero. The pressure P is $-\rho\partial\Phi/\partial t$, where ρ is the density; this is a positive constant when $r<a$, but zero when $r>a$. Then we can take the subsidiary equations to be

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial r^2}-\frac{\sigma^2}{c^2}\right)(r\Phi)&=\frac{\sigma}{c^2}r \quad r<a\\ &=0 \qquad r>a.\end{aligned}\right\} \quad\text{..................(2)}$$

The pressure must remain finite at the centre. Hence

$$r\Phi=-\frac{r}{\sigma}+A\sinh\frac{\sigma r}{c}\quad r<a \quad\text{..................(3)}$$

$$=B\exp(-\sigma r/c)\qquad r>a. \quad\text{..................(4)}$$

There can be no term in $\exp(\sigma r/c)$ in (4) since it would correspond to a wave travelling inwards. Now the pressure and the radial velocity must be continuous at $r=a$; hence Φ and $\partial\Phi/\partial r$ must be continuous. These give

$$-\frac{a}{\sigma}+A\sinh\frac{\sigma a}{c}=B\exp\left(-\frac{\sigma a}{c}\right), \quad\text{..................(5)}$$

$$-\frac{1}{\sigma}+\frac{\sigma}{c}A\cosh\frac{\sigma a}{c}=-\frac{\sigma}{c}B\exp\left(-\frac{\sigma a}{c}\right), \quad\text{............(6)}$$

whence

$$A=\frac{1}{\sigma^2}(c+a\sigma)\,e^{-\sigma a/c};\; B=\frac{1}{2\sigma^2}(c-a\sigma)\,e^{\sigma a/c}-\frac{1}{2\sigma^2}(c+a\sigma)\,e^{-\sigma a/c}. \quad (7)$$

Thus outside the original sphere

$$r\Phi=\frac{1}{2\sigma^2}(c-a\sigma)\,e^{-\sigma(r-a)/c}-\frac{1}{2\sigma^2}(c+a\sigma)\,e^{-\sigma(r+a)/c}. \quad\text{...(8)}$$

The associated pressure change is

$$P=-\frac{\rho}{2r}\left[\left(\frac{c}{\sigma}-a\right)e^{-\sigma(r-a)/c}-\left(\frac{c}{\sigma}+a\right)e^{-\sigma(r+a)/c}\right]. \quad\text{...(9)}$$

But
$$\left.\begin{aligned}\frac{c}{\sigma}-a&=0\text{ when }t\text{ is negative}\\&=ct-a\text{ when }t\text{ is positive.}\end{aligned}\right\}\quad\text{...............(10)}$$

Hence
$$\left(\frac{c}{\sigma}-a\right)e^{-\sigma(r-a)/c}=0\text{ when }ct<r-a$$
and
$$\begin{aligned}&=ct-(r-a)-a\\&=ct-r\text{ when }ct>r-a.\end{aligned}\quad\text{.........(11)}$$

Similarly
$$\left(\frac{c}{\sigma}+a\right)e^{-\sigma(r+a)/c}=0\text{ when }ct<r+a$$
and
$$=ct-r\text{ when }ct>r+a.\quad\text{.........(12)}$$

Hence the pressure disturbance is zero up to time $(r-a)/c$, when the first wave from the compressed region arrives, and after $(r+a)/c$, when the last wave passes. At intermediate times it is equal to $\frac{\rho}{2r}(r-ct)$. Thus it is equal to $\rho a/2r$ when the first wave comes, $-\rho a/2r$ when the last leaves, and varies linearly with the time in between. The compression in front of the shock is associated with an equal rarefaction in the rear*.

Within the sphere the pressure is

$$\frac{\rho\sigma}{r}\left[\frac{r}{\sigma}-\frac{1}{\sigma^2}(c+a\sigma)\sinh\frac{\sigma r}{c}\,e^{-\sigma a/c}\right]$$

$$=\rho\left[1-\frac{1}{2r}\left(\frac{c}{\sigma}+a\right)\left(e^{-\sigma(a-r)/c}-e^{-\sigma(a+r)/c}\right)\right]. \quad\text{...(13)}$$

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* Cf. Stokes, *Phil. Mag.* 34, 1849, 52.

This is equal to ρ up to time $(a-r)/c$, then drops suddenly to $\rho(1-a/2r)$, decreases linearly with the time till it reaches $-\rho a/2r$ at time $(a+r)/c$, and then rises suddenly to zero. The infinity in the pressure at the centre is only instantaneous, for the time the disturbance lasts at a given place is $2r/c$, which vanishes at the centre. It is due to the simultaneous arrival of the waves from all points on the surface of the sphere; at other points the waves from different parts of the surface arrive at different times, giving a finite disturbance of pressure over a finite interval. If $r<\frac{1}{2}a$, the pressure becomes negative immediately on the arrival of the disturbance.

The behaviour of the velocity at distant points is similar to that of the pressure. If u is the radial velocity,

$$\begin{aligned} u &= \partial\Phi/\partial r \\ &= -\sigma\Phi-\Phi/r. \qquad \text{.........................(14)} \end{aligned}$$

If r is great the first term is proportional to $1/r$, the second to $1/r^2$. The first is therefore the more important. But the first term is simply a multiple of the pressure. Hence there is no motion at a point until time $(r-a)/c$, when the matter suddenly begins to move out with velocity $a/2r$. This velocity decreases linearly until time $(r+a)/c$, when it is $-a/2r$, and then suddenly ceases. The total outward displacement contributed is zero. The second term, however, gives a small velocity which vanishes at the beginning and end of the shock, and reaches a positive maximum at time r/c. It produces a residual displacement, of order a/r times the greatest given by the first term; this represents the fact that the matter originally compressed expands till it reaches normal pressure, and the surrounding matter moves outwards to make room for it.

6.21. Consider next the analogous problem with cylindrical symmetry. With analogous initial conditions, the subsidiary equation is

$$\left.\begin{aligned} \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\Phi}{\partial\varpi}\right)-\frac{\sigma^2}{c^2}\Phi &= \frac{\sigma}{c^2} \quad \varpi<a \\ &= 0 \quad \varpi>a. \end{aligned}\right\} \qquad \text{...............(1)}$$

The solutions are Bessel functions of imaginary argument and order zero, $I_0(\sigma\varpi/c)$ and $K_0(\sigma\varpi/c)$. The latter is inadmissible within the cylinder, because it is infinite when $\varpi=0$. The former cannot occur outside it, for the following reason. The interpretation is to be an integral along a route through values of the variable with positive real parts, and when ϖ is great the asymptotic expansion of $I_0(\kappa\varpi/c)$ contains $\exp(\kappa\varpi/c)$ as a factor. Hence the solution would involve $\exp(\sigma\varpi/c)$

and therefore a disturbance travelling inwards. The solutions are therefore

$$\left.\begin{aligned}\Phi &= -\frac{1}{\sigma} + AI_0\left(\frac{\sigma\varpi}{c}\right) && 0 < \varpi < a\\ &= BK_0\left(\frac{\sigma\varpi}{c}\right) && a < \varpi.\end{aligned}\right\} \quad \dots\dots\dots\dots\dots(2)$$

Also $\partial\Phi/\partial t$ and $\partial\Phi/\partial\varpi$ must be continuous at $\varpi = a$. Hence

$$AI_0\left(\frac{\sigma a}{c}\right) - \frac{1}{\sigma} = BK_0\left(\frac{\sigma a}{c}\right), \quad \dots\dots\dots\dots\dots(3)$$

$$AI_0'\left(\frac{\sigma a}{c}\right) \quad = BK_0'\left(\frac{\sigma a}{c}\right). \quad \dots\dots\dots\dots\dots(4)$$

Also we have the identity

$$I_0'(z)\,K_0(z) - I_0(z)\,K_0'(z) = 1/z. \quad \dots\dots\dots\dots\dots(5)$$

Hence we find for points outside $\varpi = a$

$$\begin{aligned}\Phi &= -\frac{a}{c} I_0'\left(\frac{\sigma a}{c}\right) K_0\left(\frac{\sigma\varpi}{c}\right)\\ &= -\frac{1}{2\pi\iota}\frac{a}{c}\int_L I_0'\left(\frac{\kappa a}{c}\right) K_0\left(\frac{\kappa\varpi}{c}\right) e^{\kappa t}\frac{d\kappa}{\kappa}. \quad \dots\dots\dots(6)\end{aligned}$$

But*

$$I_0(z) = \frac{1}{\pi}\int_0^\pi \exp(z\cos\theta)\,d\theta, \quad \dots\dots\dots\dots\dots(7)$$

$$K_0(z) = \int_0^\infty \exp(-z\cosh v)\,dv, \quad \dots\dots\dots\dots\dots(8)$$

whence

$$\Phi = -\frac{a}{2\pi^2\iota c}\int_L\int_0^\pi\int_0^\infty \frac{1}{\kappa}\exp\kappa\left(t + \frac{a}{c}\cos\theta - \frac{\varpi}{c}\cosh v\right)\cos\theta\,d\kappa\,d\theta\,dv. \quad \dots\dots(9)$$

Performing first the integration with regard to κ, we obtain a function of the form $H(t-b)$. Thus

$$\Phi = -\frac{a}{\pi c}\int_0^\pi\int_0^\infty \cos\theta\,d\theta\,dv \quad \dots\dots\dots\dots\dots(10)$$

where the range of integration is restricted by the condition that

$$ct + a\cos\theta - \varpi\cosh v > 0. \quad \dots\dots\dots\dots\dots(11)$$

It follows at once that there is no movement at a place until time $(\varpi - a)/c$. Integrating next with regard to v, we see that the admissible values range from 0 to $\cosh^{-1}\{(ct + a\cos\theta)/\varpi\}$, provided the quantity in the parentheses is greater than unity. Hence

$$\Phi = -\frac{a}{\pi c}\int\cos\theta\cosh^{-1}\left(\frac{ct + a\cos\theta}{\varpi}\right)d\theta. \quad \dots\dots\dots(12)$$

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* The notation is that of Watson's *Bessel Functions*.

So long as $ct > \varpi - a$, there will be a finite range of integration with regard to θ; the lower limit is then always zero. If $ct > \varpi + a$, the inequality is satisfied for all values of θ up to π, and therefore π is the upper limit. If $ct < \varpi + a$, the inequality is not satisfied when $\theta = \pi$, and the upper limit is $\cos^{-1}(\varpi - ct)/a$. The disturbance at any point may therefore be divided into three stages, the first until $ct = \varpi - a$, the second from then till $ct = \varpi + a$, and the third later.

We are concerned chiefly with the pressure. This remains constant until $ct = \varpi - a$; in the next stage it is equal to

$$P = \frac{\rho a}{\pi} \int_0^{\cos^{-1}(\varpi - ct)/a} \frac{\cos\theta \, d\theta}{\{(ct + a\cos\theta)^2 - \varpi^2\}^{\frac{1}{2}}} \quad \text{............(13)}$$

and in the last to a similar integral with the upper limit replaced by π. By applying the transformation

$$ct + a - \varpi = 2b; \quad ct + a\cos\theta - \varpi = 2b\cos^2\psi \quad \text{.........(14)}$$

and integrating on the supposition that b is small, we find that soon after the arrival of the wave

$$P = \tfrac{1}{2}\rho \sqrt{\left(\frac{a}{\varpi}\right)}\left[1 - \frac{3}{8}\frac{b}{a} - \frac{1}{4}\frac{b}{\varpi}\right]. \quad \text{...............(15)}$$

When the wave arrives the pressure therefore jumps to $\frac{1}{2}\rho\,(a/\varpi)^{\frac{1}{2}}$, and then falls by $\frac{3}{8}\,c/a$ of itself per unit time. The corresponding fraction in the spherical problem is c/a. At time ϖ/c the pressure is still positive; but when $ct = \varpi + a$ or more and θ is greater than $\frac{1}{2}\pi$, the integrand in (13) is numerically greater than for the supplementary value of θ, and thus P is negative. The passage of the wave of rarefaction is therefore indefinitely protracted. To find out how it dies down with the time let us suppose that ct is greater than ϖ, and that a is small compared with either. Then

$$P = -\tfrac{1}{2}\rho \frac{a^2 ct}{(c^2t^2 - \varpi^2)^{\frac{3}{2}}} \quad \text{........................(16)}$$

approximately. The residual disturbance falls off like t^{-2}.

6.22. If the motion was one-dimensional, as for instance if the original excess of pressure was confined to a length $2a$ of a tube, the resulting disturbance of pressure would consist of two waves, each with an excess of pressure equal to $\frac{1}{2}\rho$, travelling out in opposite directions with velocity c. Comparing the results for the three cases, we see that the first disturbance at a given point outside the region originally disturbed, in each case at the same distance from the nearest point of it, occurs in each case at the same moment. The increase of pressure in the one-dimensional problem is $\frac{1}{2}\rho$, in the two-dimensional one $\frac{1}{2}\rho(a/\varpi)^{\frac{1}{2}}$, and in the three-dimensional one $\frac{1}{2}\rho(a/r)$. In the first case the pressure remains constant for time $2a/c$, and then drops to zero and remains there. In the cylindrical problem it begins to fall instantly, and becomes negative in an interval less than $2a/c$; it then reaches a negative maximum, and dies down again asymptotically to zero. In the spherical one it decreases linearly with the time and reaches a negative value equal to the original positive one at time $2a/c$; then it suddenly becomes zero again.

6.3. *Diverging waves produced by a sphere oscillating radially**. Suppose that a sphere of radius a begins at time 0 to oscillate radially in period $2\pi/n$. We require the motion of the air outside it.

The velocity potential Φ satisfies the equation

$$\frac{\partial^2}{\partial r^2}(r\Phi) - \frac{\partial^2}{c^2\partial t^2}(r\Phi) = 0. \qquad \text{.....................(1)}$$

Initially all is at rest; the solution is therefore

$$r\Phi = Ae^{-\sigma r/c}. \qquad \text{..............................(2)}$$

When $r = a$ the outward displacement is, say, $\frac{1}{n}\sin nt$ when $t > 0$, and the outward velocity $\cos nt$. Hence

$$A\left[\frac{\partial}{\partial r}\left(\frac{1}{r}e^{-\sigma r/c}\right)\right]_{r=a} = \frac{\sigma^2}{\sigma^2+n^2} \qquad \text{.....................(3)}$$

and

$$r\Phi = -\frac{\sigma^2 a^2}{(\sigma^2+n^2)(1+\sigma a/c)}\exp\left\{-\frac{\sigma}{c}(r-a)\right\}$$

$$= -\frac{c^2a^2}{c^2+a^2n^2}\exp\left\{-\frac{\sigma}{c}(r-a)\right\}\left\{\cos nt + \frac{na}{c}\sin nt - \exp\left(-\frac{ct}{a}\right)\right\}$$

$$= -\frac{c^2a^2}{c^2+a^2n^2}\left[\cos n\left\{t-\frac{r-a}{c}\right\} + \frac{na}{c}\sin n\left\{t-\frac{r-a}{c}\right\} - \exp\left\{-\frac{ct-r+a}{a}\right\}\right] \qquad \text{...............(4)}$$

when $ct > r - a$.

The solution has a periodic part with a period equal to that of the given disturbance, together with a part dying down with the time at a rate independent of n, but involving the size of the sphere. As there is no corresponding term in the problem of 6.2 we may regard it as a result of the constraint introduced by the presence of the rigid sphere. Its effect on the velocity or the pressure is to that of the second term in a ratio comparable with $(c/na)^2$.

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* Love, *Proc. Lond. Math. Soc.* (2) 2 1904, 88; Bromwich, *ib.* (2) 15, 1916, 431.

6.4. A spherical thermometer bulb is initially at a uniform temperature equal to that of its surroundings. The temperature of the air decreases with height, and the thermometer is carried upwards at such a rate that the temperature at the outside of the glass varies linearly with the time. Find how the mean temperature of the mercury varies*.

The temperature within the bulb satisfies the equation

$$\frac{\partial^2}{\partial r^2}(rV) - q^2 rV = 0, \quad \dots\dots(1)$$

where

$$\sigma = h^2 q^2. \quad \dots\dots(2)$$

That at the outer surface of the glass is Gt, where G is a constant. But the glass has only a finite conductivity, so that the surface condition at the outside of the mercury is

$$\frac{\partial V}{\partial r} = K(Gt - V), \quad \dots\dots(3)$$

K being another constant. The solution of (1) is

$$V = \frac{A}{r}\sinh qr \quad \dots\dots(4)$$

and (3) gives

$$A = \frac{a^2 KG/\sigma}{Ka \sinh qa + qa \cosh qa - \sinh qa} \quad \dots\dots(5)$$

where a is the inner radius of the glass.

The mean temperature within the bulb is

$$\begin{aligned} V_0 &= \frac{3}{a^3}\int_0^a r^2 V\,dr \\ &= \frac{3A}{a^3 q^2}(qa \cosh qa - \sinh qa) \\ &= \frac{3KG}{a\sigma q^2}\,\frac{qa \cosh qa - \sinh qa}{Ka \sinh qa + qa \cosh qa - \sinh qa}. \quad \dots\dots(6) \end{aligned}$$

In applying the partial-fraction rule, we notice that near $\sigma = 0$

$$\begin{aligned} V_0 &= \frac{3KG}{a\sigma q^2}\,\frac{(qa)^3\left(\frac{1}{3} + \frac{1}{30}q^2a^2\right)}{qa\left(Ka + \frac{1}{6}Kq^2a^3 + \frac{1}{3}q^2a^2\right)} \\ &= G\left\{\frac{1}{\sigma} - \left(\frac{1}{15}\frac{a^2}{h^2} + \frac{1}{3}\frac{a}{h^2K}\right)\right\} \\ &= G\left\{t - \frac{a^2}{h^2}\left(\frac{1}{15} + \frac{1}{3aK}\right)\right\}, \end{aligned}$$

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* Bromwich, *Phil. Mag.* 37, 1919, 407–419; A. R. McLeod, *Phil. Mag.* 37, 1919, 134.

so that there is a constant lag in the temperature of the mercury in comparison with that of the air.

The other zeros of the denominator give exponential contributions, which are evaluated in Bromwich's paper.

6.5. A cylinder of internal radius a can rotate freely about its axis. It is filled with viscous liquid, and the whole is rotating as if solid with angular velocity ω_0. The cylinder is instantaneously brought to rest at time $t=0$, and immediately released. Find the angular velocity later. (Math. Trip. Schedule B, 1926.)

The motion is two-dimensional, and there is a stream-function ψ satisfying the equation

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} \quad \ldots\ldots\ldots\ldots(1)$$

where ν is the kinematic viscosity. Since the motion is symmetrical about an axis the right side is identically zero. Put

$$\frac{\partial}{\partial t} = \sigma, \quad \sigma = \nu r^2, \quad \nabla^2 = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial}{\partial\varpi}\right). \quad \ldots\ldots\ldots\ldots(2)$$

Initially
$$\psi = \tfrac{1}{2}\varpi^2\omega_0; \quad \nabla^2\psi = 2\omega_0 \quad \ldots\ldots\ldots\ldots(3)$$

and the subsidiary equation is

$$(\nabla^2 - r^2)\nabla^2\psi = -2r^2\omega_0. \quad \ldots\ldots\ldots\ldots(4)$$

The solution is

$$\psi = AI_0(r\varpi) + BK_0(r\varpi) + C\log\varpi + D + \tfrac{1}{2}\varpi^2\omega_0. \quad \ldots\ldots(5)$$

The velocity must be finite on the axis; hence B and C are zero. Also D is independent of ϖ and therefore cannot affect the motion.

If I be the moment of inertia of the cylinder per unit length and ω its angular velocity, the equation of motion of the cylinder is

$$I\frac{\partial\omega}{\partial t} = -2\pi a p_s, \quad \ldots\ldots\ldots\ldots(6)$$

where p_s is the shearing stress in the fluid. Now

$$\begin{aligned} p_s &= \nu\rho\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) \\ &= \nu\rho a\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) \quad \ldots\ldots\ldots\ldots(7) \end{aligned}$$

evaluated on the outer boundary at the point $(a, 0)$. Hence

$$p_s = \nu\rho\frac{A}{a}r\left[raI_0(ra) - 2I_0'(ra)\right]. \quad \ldots\ldots\ldots\ldots(8)$$

Since the cylinder starts from rest,

$$I\sigma\omega = -2\pi\nu\rho\, Ar\,[ra\,I_0(ra) - 2I_0'(ra)]. \quad \text{............(9)}$$

Also $a\omega$ must be equal to the velocity of the fluid. Hence

$$a\omega = Ar\,I_0'(ra) + a\omega_0. \quad \text{........................(10)}$$

Eliminating A we have

$$\omega\,[(K\sigma - 2)\,I_0'(ra) + ra\,I_0(ra)] = \omega_0\,[ra\,I_0(ra) - 2I_0'(ra)] \text{...(11)}$$

where $$2\pi\nu\rho a\,K = I. \quad \text{........................(12)}$$

The operational solution is therefore

$$\omega = \omega_0 \frac{ra\,I_0(ra) - 2I_0'(ra)}{(K\sigma - 2)\,I_0'(ra) + ra\,I_0(ra)}. \quad \text{............(13)}$$

But

$$I_0(ra) = 1 + \tfrac{1}{4}r^2a^2 + \tfrac{1}{64}r^4a^4 + \dots;\quad I_0'(ra) = \tfrac{1}{2}ra + \tfrac{1}{16}r^3a^3 + \dots. \quad (14)$$

The contribution from $\sigma = 0$ is found to be

$$\frac{\omega_0 a^2}{a^2 + \nu K} = \omega_1 \quad \text{........................(15)}$$

say. This is the ultimate angular velocity. The other terms arise from the zeros of the denominator. If we write ιk for r, and substitute for K in terms of ω_1, we find that these satisfy

$$\left\{(\omega_0 - \omega_1)\frac{a^2k^2}{4\omega_1} + 2\right\} J_1(ka) - ka\,J_0(ka) = 0.$$

The solution is then of the form $\omega_1 + \Sigma A_k e^{-k^2\nu t}$. The coefficients A_k are determinable by the usual method.

CHAPTER VII

DISPERSION

In the propagation of sound waves in air and of waves on strings the velocity of travel of waves is independent of the period. In many problems this is not the case; waves on water afford an important example.

7.1. Consider a layer of incompressible fluid of density ρ and depth H. Take the origin in the undisturbed position of the free surface, the axis of z upwards, and those of x and y in the horizontal plane. Put

$$\frac{\partial}{\partial t} = \sigma;\ \frac{\partial}{\partial x} = p;\ \frac{\partial}{\partial y} = q;\ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = r^2. \quad \text{..............(1)}$$

The velocity potential Φ satisfies the equation

$$\frac{\partial^2\Phi}{\partial z^2}+r^2\Phi=0. \quad \text{.............................(2)}$$

At the bottom the vertical velocity vanishes. At the free surface, so long as the motion is only slightly disturbed from rest, we have

$$\frac{\partial\Phi}{\partial z}=\frac{\partial\zeta}{\partial t}, \quad \text{.............................(3)}$$

where ζ is the elevation of the free surface. Then we must have

$$\Phi=-\frac{\sigma}{r}\frac{\cos r(z+H)}{\sin rH}\zeta. \quad \text{.....................(4)}$$

The pressure just under the free surface is $-T\rho r^2\zeta$, where $T\rho$ is the surface tension.

But by Bernoulli's equation it is also equal to $-\rho\sigma\Phi-g\rho\zeta+F(t)$, and $F(t)$ does not affect the motion and can therefore be omitted. We have therefore the further surface condition

$$-\sigma\Phi=(g-Tr^2)\zeta. \quad \text{.......................(5)}$$

Combining this with (4) we have the differential equation for ζ

$$\{\sigma^2-(g-Tr^2)\,r\tan rH\}\,\zeta=0. \quad \text{..................(6)}$$

If the fluid starts from rest with ζ equal to ζ_0, a known function of x and y, the subsidiary equation will have a term $\sigma^2\zeta_0$ on the right, and the operational solution will be

$$\zeta=\frac{\sigma^2}{\sigma^2-(g-Tr^2)\,r\tan rH}\zeta_0 \quad \text{.......................(7)}$$

$$=\cosh t\{(g-Tr^2)\,r\tan rH\}^{\frac{1}{2}}.\zeta_0. \quad \text{..................(8)}$$

In the corresponding problem of a uniform string in 4.1 the coefficient of t was simply pc.

Suppose first that the original disturbance consists of an infinite elevation along the axis of y, with no disturbance of the surface anywhere else. Then ζ_0 can be replaced by p, and (8) is equivalent to the integral

$$\zeta=\frac{1}{2\pi\iota}\int_L e^{\kappa x}\cosh t\{(g-T\kappa^2)\,\kappa\tan\kappa H\}^{\frac{1}{2}}\,d\kappa. \quad \text{.........(9)}$$

The integrand has an essential singularity wherever κH is an odd multiple of $\frac{1}{2}\pi$; L cannot therefore cross the real axis within a finite distance of the origin, but becomes two branches extending to $+\infty$ above and below the axis.

7.2. *The method of steepest descents.* The elevation of the surface is thus expressed in terms of integrals of the type

$$S=\int_A^B \exp t\{f(z)\}\,dz, \quad\text{.........................}(1)$$

where $f(z)$ is an analytic function, and t is real, positive, and independent of z. Put, following Debye,

$$f(z)=R+\iota I, \quad\text{..............................}(2)$$

thus expressing it in real and imaginary parts. If the integral is taken along an arbitrary path, the integrand will be the product of a variable positive factor with one whose absolute value is unity, but which varies in argument more and more rapidly the greater t is. There will evidently be advantages in choosing the path in such a way that the large values of R are concentrated in the shortest possible interval on it. Now if

$$z=x+\iota y, \quad\text{................................}(3)$$

we shall have
$$\frac{\partial^2 R}{\partial x^2}+\frac{\partial^2 R}{\partial y^2}=0;\ \frac{\partial^2 I}{\partial x^2}+\frac{\partial^2 I}{\partial y^2}=0. \quad\text{.................}(4)$$

It follows that R can never be an absolute maximum. But it can have stationary points, where

$$\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=0, \quad\text{..............................}(5)$$

and we know that these points will also be stationary points of I and zeros of $f'(z)$. These points are usually called the 'saddle points,' or sometimes 'cols.' Through any saddle point it will in general be possible to draw two (sometimes more) curves such that R is constant along them. In sectors between these curves R will be alternately greater and less than at the saddle point itself. The sectors where R is greater may be called the 'hills,' those where it is less the 'valleys.' If our path of integration is to be chosen so as to avoid large values of R, it must avoid the hills, and keep as far as possible to the valleys. If then the complex plane is marked out by the lines of R constant through all the saddle points, and A and B lie within the same valley, our path must never go outside this valley; but if A and B lie in different valleys, the passage from one valley to another must take place through a saddle point. In the latter case the value of the integral will be much greater than in the former, and therefore interest attaches chiefly to the case where the limits of the integral lie in different valleys.

The paths actually chosen are specified rather more narrowly; the direction of the path at any point is chosen so that $|\partial R/\partial s|$ is as great as possible. If ψ is the inclination of the tangent to the path to the axis of x, we have

$$\frac{\partial R}{\partial s} = \cos\psi \frac{\partial R}{\partial x} + \sin\psi \frac{\partial R}{\partial y}, \quad \text{......................(6)}$$

and if this is to be a numerical maximum for variations in ψ

$$0 = -\sin\psi \frac{\partial R}{\partial x} + \cos\psi \frac{\partial R}{\partial y}$$

$$= \frac{\partial R}{\partial n} = \frac{\partial I}{\partial s}, \quad \text{......................................(7)}$$

where dn is an element of length normal to the path, drawn so that when ds is in the direction of x increasing, dn is in the direction of y increasing. Hence I is constant along the path. Such a path is called a 'line of steepest descent.' There will be one in each valley. In general the limits of the integral will not themselves lie on lines of steepest descent, but can be joined to them by paths within the valleys.

In general lines of R constant through different saddle points will not intersect; and there will be only one saddle point on each line of steepest descent. For the former event would imply that R has the same value at two saddle points, the latter that I has, and either of these events will be exceptional. It follows that as we proceed along a line of steepest descent R will rarely reach a minimum and then proceed to increase again. For if R had a minimum $\partial R/\partial s$ would be zero; but $\partial I/\partial s$ is zero by construction, and therefore the point would be another saddle point. Lines of steepest descent usually terminate only at singularities of $f(z)$ or at infinity.

The path of integration once chosen, the greater t is the more closely the higher values of the integrand will be concentrated about the saddle points. Thus we can obtain an approximation, which will be better the larger t is, by considering only the parts of the path in these regions. In these conditions we can take

$$f(z) = f(z_0) + \tfrac{1}{2}(z - z_0)^2 f''(z_0), \quad \text{...................(8)}$$

where z_0 is a saddle point. Put

$$|f''(z_0)| = A; \quad |z - z_0| = r. \quad \text{......................(9)}$$

Then on a line of steepest descent

$$f(z) = f(z_0) - \tfrac{1}{2}Ar^2, \quad \text{.........................(10)}$$

and we can put

$$\arg(z - z_0) = \alpha, \quad \text{............................(11)}$$

on the side after passing through z_0. Then

$$S = 2\int_0^\infty \exp\{t f(z_0)\} \exp(-\tfrac{1}{2}Atr^2)\, d(r \exp \iota\alpha)$$

$$= \left(\frac{2\pi}{At}\right)^{\frac{1}{2}} \exp\{t f(z_0) + \iota\alpha\}. \quad \ldots\ldots(12)$$

To get from A to B it may be necessary to pass through two or more saddle points, with probably traverses in the valleys between the lines of steepest descent. Then each saddle point will make its contribution to the integral.

The error involved in this approximation arises from the terms of the third and higher orders omitted from (8). Its accuracy therefore depends on $\exp(-\frac{1}{2}tAr^2)$ having become small before $\exp(-\frac{1}{6}tr^3|f'''(z_0)|)$ has begun to differ appreciably from unity. Hence

$$\tfrac{1}{2}tA\left[\frac{6}{t\,|f'''(z_0)|}\right]^{\frac{2}{3}} = 1{\cdot}6\, t^{\frac{1}{3}} \frac{|f''(z_0)|}{|f'''(z_0)|^{\frac{2}{3}}} \quad \ldots\ldots(13)$$

must be large. In most cases the approximation is asymptotic, and does not represent the first term of a convergent series.

7.3. In problems of wave motion we often have to evaluate integrals of the form

$$\zeta = \frac{1}{2\pi\iota}\int_L \phi(\kappa) \exp(\kappa x - \gamma t)\, d\kappa, \quad \ldots\ldots(1)$$

where $\phi(\kappa)$ and γ are known functions of κ. As a rule $\phi(\kappa)$ is an even function, and γ an odd one. When κ is purely imaginary γ is also purely imaginary. We require the motion for large values of t, and possibly also of x.

The function $\phi(\kappa)$ usually introduces no difficulty. It does not involve x or t, and therefore when these are large enough it can be treated as constant throughout the range where the integrand is appreciable.

It is usually convenient to replace κ by $\iota\kappa$ and γ by $\iota\gamma$, and to consider the equivalent integral of the form

$$\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\kappa) \exp \iota(\kappa x - \gamma t)\, d\kappa. \quad \ldots\ldots(2)$$

The saddle points are given by

$$x - \gamma' t = 0 \quad \ldots\ldots(3)$$

so that a given ratio x/t specifies a set of predominant values of κ_0. But γ' is an even function of κ, and therefore if κ_0 is a saddle point, $-\kappa_0$ will be another, and if the adopted path passes through either it will pass through the other. We may take κ_0 real and positive. Also

$$f''(\kappa_0) = -\iota\gamma_0'' t. \qquad\text{.........................(4)}$$

Thus if γ_0'' is positive $\kappa - \kappa_0$ has argument $-\frac{1}{4}\pi$ on the line of steepest descent, and the contribution from κ_0 is

$$\frac{1}{2\pi}\left(\frac{2\pi}{|\gamma_0'' t|}\right)^{\frac{1}{2}} e^{-\frac{1}{4}\iota\pi} \exp \iota(\kappa_0 x - \gamma_0 t). \qquad\text{..................(5)}$$

The contribution from $-\kappa_0$ (where γ'' is negative) is similarly

$$\frac{1}{2\pi}\left(\frac{2\pi}{|\gamma_0'' t|}\right)^{\frac{1}{2}} \psi(\kappa_0)\, e^{\frac{1}{4}\iota\pi} \exp - \iota(\kappa_0 x - \gamma_0 t) \qquad\text{...........(6)}$$

and the two together give

$$\left(\frac{2}{\pi|\gamma_0'' t|}\right)^{\frac{1}{2}} \psi(\kappa_0) \cos(\kappa_0 x - \gamma_0 t - \tfrac{1}{4}\pi). \qquad\text{...............(7)}$$

Similarly if γ_0'' is negative the two saddle points give

$$\left(\frac{2}{\pi|\gamma_0'' t|}\right)^{\frac{1}{2}} \psi(\kappa_0) \cos(\kappa_0 x - \gamma_0 t + \tfrac{1}{4}\pi). \qquad\text{..............(8)}$$

7.31. These formulae, due to Kelvin, are the fundamental ones of the theory of dispersion. Consider first the cosine factor, and suppose x increased by δx and t by δt. Then $\kappa_0 x - \gamma_0 t$ is increased by

$$\kappa_0 \delta x - \gamma_0 \delta t + (x - \gamma_0' t)\,\delta\kappa_0,$$

the term in $\delta\kappa_0$ appearing because κ_0 is defined as a function of x and t by (3). But the coefficient of $\delta\kappa_0$ is zero by (3). If then t is kept constant, ζ will vary with x with period $2\pi/\kappa_0$; and if x is kept constant, ζ will vary with t with period $2\pi/\gamma_0$. Hence $2\pi/\kappa_0$ is the wave-length, and $2\pi/\gamma_0$ the period, of the waves passing a given place. A phase occurring at a given place and time is reproduced after an interval δt at a place such that $\delta x = \gamma_0\, \delta t/\kappa_0$. Hence γ_0/κ_0 is the velocity of travel of individual waves. It may be denoted by c, and called the wave-velocity.

But κ_0 has been defined by the equation

$$x - \gamma_0' t = 0$$

so that a given wave-length and period always occur when x/t has a particular value; they seem to travel out with velocity γ_0', which is called the group-velocity. It may also be denoted by C. In general the wave-velocity and the group-velocity are unequal, so that a given wave changes in period and length as it progresses. They are evidently connected by the relation

$$\frac{\partial}{\partial\kappa}(\kappa c) = C. \qquad\text{.............................(9)}$$

7.4. Returning now to 7.1 (9) we can separate the hyperbolic function into two exponentials, which will represent wave-systems travelling out in opposite directions. One of them is equivalent to

$$\zeta = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp \iota(\kappa x - \gamma t)\, d\kappa, \quad \text{..................(1)}$$

where

$$\gamma^2 = (g + T\kappa^2)\,\kappa \tanh \kappa H. \quad \text{.....................(2)}$$

When κ is small,

$$c = (gH)^{\frac{1}{2}}\left(1 - \tfrac{1}{6}\kappa^2 H^2 + \frac{1}{2}\frac{T\kappa^2}{g}\right), \quad \text{..................(3)}$$

$$C = (gH)^{\frac{1}{2}}\left(1 - \tfrac{1}{2}\kappa^2 H^2 + \frac{3}{2}\frac{T\kappa^2}{g}\right) \text{..................(4)}$$

When κ is great,

$$c = (T\kappa)^{\frac{1}{2}};\ C = \tfrac{3}{2}(T\kappa)^{\frac{1}{2}}. \quad \text{........................(5)}$$

In all ordinary cases T/g is insignificant in comparison with H^2. Hence for some intermediate value of κ the group-velocity is a minimum; it tends to infinity for very short waves, and to a finite limit for very long ones. Three cases therefore arise. If x/t is less than the minimum group-velocity, there will be no saddle-point on the real axis, and the disturbance will be small*. If it lies between this minimum and $(gH)^{\frac{1}{2}}$, two (positive) values of κ will give saddle-points, and each will contribute to the motion. If it is greater than $(gH)^{\frac{1}{2}}$, the only saddle-point will correspond to a short wave. The disturbance at a given point will therefore be in three stages. In the first, leading up to time $x/(gH)^{\frac{1}{2}}$, only very short capillary waves will occur. Then long gravity waves will arrive, the wave-lengths of those reaching the point diminishing as time goes on. Superposed on them are further capillary waves, their length increasing with the time. At a certain moment the wave-lengths of the two sets become equal. This corresponds to the arrival of the waves with the minimum group-velocity. From then on the water is smooth.

7.41. Two typical cases therefore arise according as the wave-length is large or small compared with the one that gives the minimum group-velocity. Take first gravity waves, such that x/t is small compared with

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* It can be shown that the saddle-points are so placed that the relevant one contributes an exponential with a negative index to the solution. This is almost obvious from considerations of energy.

$(gH)^{\frac{1}{2}}$ but large compared with the minimum group-velocity. In these conditions we can write simply

$$\gamma^2 = g\kappa;\ c = (g/\kappa)^{\frac{1}{2}};\ C = \tfrac{1}{2}(g/\kappa)^{\frac{1}{2}};\ dC/d\kappa = -\tfrac{1}{4}(g/\kappa^3)^{\frac{1}{2}}. \quad \text{......}(1)$$

The solution is then

$$\left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} 2 \left(\frac{\kappa_0{}^3}{g}\right)^{\frac{1}{4}} \cos\left(\kappa_0 x - \gamma_0 t + \tfrac{1}{4}\pi\right) \quad \text{..............}(2)$$

where

$$\frac{x}{t} = \frac{1}{2}\left(\frac{g}{\kappa_0}\right)^{\frac{1}{2}} \quad \text{..........................}(3)$$

and the amplitude increases towards the rear of the wave train like $x^{-\frac{3}{2}}$.

7.42. Take next the capillary waves, short enough for gravity to be neglected. Then

$$\gamma^2 = T\kappa^3;\ c = (T\kappa)^{\frac{1}{2}};\ C = \tfrac{3}{2}(T\kappa)^{\frac{1}{2}};\ dC/d\kappa = \tfrac{3}{4}(T/\kappa)^{\frac{1}{2}}. \quad \text{......}(1)$$

At a given instant the amplitude is therefore proportional to $\kappa_0{}^{\frac{1}{4}}$, or to $x^{\frac{1}{2}}$. The front of the disturbance is therefore composed of a series of capillary waves whose amplitude tends to infinity, and the time taken for them to arrive is infinitesimal.

This impossible result arises from the form assumed for the original displacement. In taking $\zeta_0 = p$ we assumed that unit volume of liquid was originally released on unit length of the actual line $x = 0$ in the surface. The mean height of this mass of liquid was therefore infinite, and its potential energy also infinite. The system being frictionless, this energy must be present somewhere in the waves existing at any instant, and infinite amplitudes are therefore a natural consequence of the initial conditions. If instead we suppose that the same volume of fluid was originally raised, but that it was distributed uniformly between $x = \pm l$, its elevation was $1/2l$ in this range. Expressed in operational form this gives

$$\zeta_0 = \frac{1}{2l}[H(x+l) - H(x-l)] = \frac{1}{2l}(e^{pl} - e^{-pl}). \text{...........}(2)$$

The appropriate solution can be found from 7.4 (1) by introducing a factor $\dfrac{1}{2l\iota\kappa}(e^{\iota\kappa l} - e^{-\iota\kappa l})$ or $\dfrac{\sin \kappa l}{\kappa l}$ into the integrand. If the solution already found makes $\kappa_0 l$ small, this additional factor will be practically unity, and the same solution will hold. Waves whose length is large compared with the extent of the original disturbance will therefore not be much affected by its finiteness.

But if $\kappa_0 l$ is large we must consider separately the contributions from the terms in $e^{\iota\kappa l}$ and $e^{-\iota\kappa l}$. The factor $1/\kappa l$ will give an extra $1/\kappa_0 l$ in all the solutions. The result will be that at a given instant the amplitude of a gravity wave will vary like $\kappa_0^{-\frac{1}{4}}$ and that of a capillary wave like $\kappa_0^{-\frac{3}{4}}$. Waves whose length is short compared with that of the original disturbance are therefore heavily reduced in amplitude.

On deep water the minimum group-velocity is 18 cm./sec., corresponding to a wave-length of 4·6 cm. and a wave-velocity of 28 cm./sec. If the original disturbance has a horizontal extent of 1 cm. or so, only waves with lengths under 1 cm. or so will be affected, and the amplitudes of both gravity and capillary waves will increase steadily with diminishing group-velocity. A wave of large amplitude will therefore bring up the rear, and will leave smooth water behind it. This is observable in the waves caused by raindrops and other very concentrated disturbances. But if the extent of the original disturbance exceeds a few centimetres the capillary waves produced will be very small, and the largest amplitude will be associated with a wave whose length is comparable with the width of the disturbed region. The largest wave produced by the splash of a brick, for instance, has a length of the order of a foot.

7.5. Two exceptional cases may arise in the treatment of dispersion, which are both illustrated in the present problem. The validity of the approximation 7.2 (12) depends on $\exp\{tf(z)\}$ being proportional to $\exp(-\frac{1}{2}Atr^2)$ on a line of steepest descent. If $f''(z)$ has varied by a considerable fraction of A before this exponential has become small the approximation will not be good. This may happen if A is itself small, or if there are two saddle-points close together. Instances occur when there is a maximum or minimum group-velocity, or if the group-velocity tends to a finite limit when κ becomes very small. In the former case a value of x/t a little greater than the minimum group-velocity will give two slightly different finite values of κ_0. In the latter κ_0 may be small enough for the proximity of $-\kappa_0$ to affect the contributions of both.

7.51. Since γ is an odd function of κ, we may suppose that when κ is small

$$\gamma = c_0\kappa - c_2\kappa^3 + O(\kappa^5) \quad \text{.........................(1)}$$

and then 7.3 (2) is equivalent to

$$\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\kappa) \exp \iota(\kappa x - c_0\kappa t + c_2\kappa^3 t)\, d\kappa. \quad \text{.........(2)}$$

There are saddle-points where

$$\kappa = \pm \left(\frac{c_0 t - x}{3c_2 t}\right)^{\frac{1}{2}}. \quad \text{......................} (3)$$

If, as in 7.4, $\psi(\kappa)$ is constant or tends to a limit different from zero when κ tends to zero, this integral is nearly

$$\zeta = \frac{1}{\pi}\int_0^\infty \psi(0) \cos\{\kappa(x - c_0 t) + c_2 \kappa^3 t\}\, d\kappa$$

$$= \frac{1}{\pi}\psi(0)\,(c_2 t)^{-\frac{1}{3}} \int_0^\infty \cos(v^3 - mv)\, dv \quad \text{..............} (4)$$

where

$$m = (c_0 t - x)/(c_2 t)^{\frac{1}{3}} \text{.........................} (5)$$

The integral involved here is called an Airy integral*. It is finite and positive, but not stationary, when $m = 0$; it has a maximum when $m = 1{\cdot}28$, and oscillates for greater values of m with steadily decreasing amplitude. For negative values of m it tends asymptotically to zero.

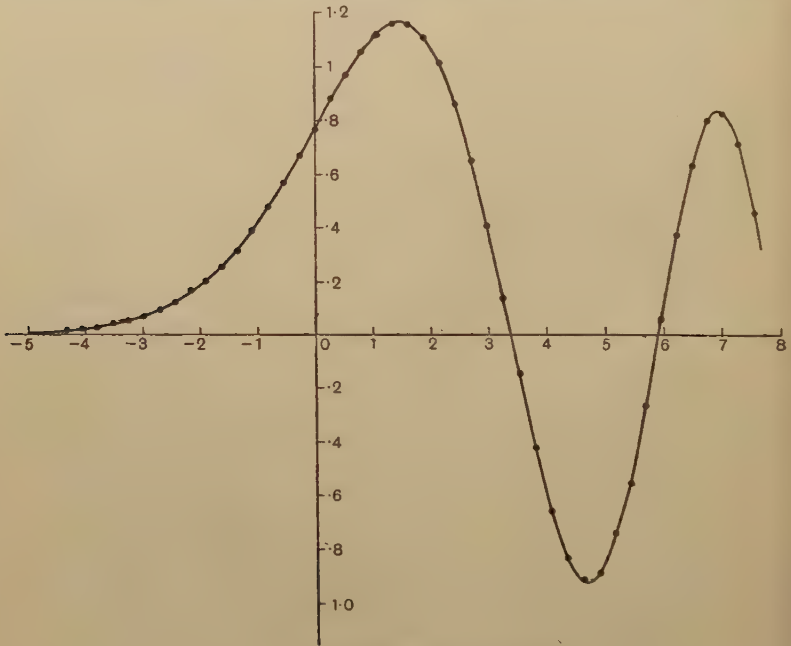

Fig. 7. Graph of the Airy integral.

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[*] Airy tabulates $\int_0^\infty \cos \frac{1}{2}\pi(v^3 - mv)\, dv$ in *Camb. Phil. Trans.* 8, 1849, 598. The graph given here is adapted from Airy's table. The integral can also be expressed in terms of Bessel functions of order $\pm\frac{1}{3}$. See Watson, 188–190.

Consequently the disturbance we are considering produces an immediate rise of the level of the water at all distances greater than $c_0 t$, though this rise is very small at great distances. The maximum rise of level is where x is rather less than $c_0 t$, so that it has not travelled out from the origin so far as a wave with the limiting velocity would. The maximum is followed by a series of waves of gradually diminishing length and amplitude, merging ultimately into gravity waves of the deep-water type.

7.52. In the case where there is a minimum group-velocity for a finite wave-length, let us, with a somewhat different notation from that used so far, denote the minimum group-velocity by γ_0' and the corresponding values of κ and γ by κ_0 and γ_0. Put

$$\kappa - \kappa_0 = \kappa_1 \quad \text{.................................}(1)$$

$$\gamma = \gamma_0 + \gamma_0' \kappa_1 + 0 + \tfrac{1}{6}\gamma_0''' \kappa_1^3 + \dots \quad \text{..................}(2)$$

Then

$$\zeta = \frac{1}{2\pi}\int_{-\kappa_0}^{\infty} \psi(\kappa) \exp \iota(\kappa_0 x - \gamma_0 t) \exp \iota(\kappa_1 x - \gamma_0' \kappa_1 t - \tfrac{1}{6}\gamma_0''' \kappa_1^3 t)\, d\kappa_1 \text{...}(3)$$

with an analogous contribution from the negative values of κ; hence with the same type of approximation as before we shall have

$$\zeta = \frac{2}{\pi}\int_0^{\infty} \psi(\kappa_0) \cos(\kappa_0 x - \gamma_0 t) \cos(\kappa_1 x - \gamma_0' \kappa_1 t - \tfrac{1}{6}\gamma_0''' \kappa_1^3 t)\, d\kappa_1$$

$$= \frac{2}{\pi}\psi(\kappa_0) \cos(\kappa_0 x - \gamma_0 t)\,(\tfrac{1}{6}\gamma_0''' t)^{-\frac{1}{3}} \int_0^{\infty} \cos(v^3 - mv)\, dv \quad \text{...}(4)$$

where

$$m = \frac{x - \gamma_0' t}{(\frac{1}{6}\gamma_0''' t)^{\frac{1}{3}}}. \quad \text{..............................}(5)$$

The solution is therefore the product of a cosine and an Airy integral, the interval between consecutive zeros being much greater for the former than for the latter. In the neighbourhood of a point that has travelled out with the minimum group-velocity the waves have the corresponding period and wave-length, but their amplitude falls off rapidly towards the rear. In front of this point the amplitude increases for a while and then oscillates.

In both these exceptional cases we notice that the amplitude associated with the critical velocity falls off only like the inverse cube root of the time, whereas in the typical case it falls off like the inverse square root. Hence the further the disturbance progresses the more will the waves with the critical group-velocities predominate in relation to the others. In waves on water, however, these phenomena are often modified by the greater damping effect of viscosity on short waves.

CHAPTER VIII

BESSEL FUNCTIONS

8.1. The Bessel functions of imaginary argument are defined by the expansions

$$I_n(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!} \quad \text{.........................(1)}$$

where $(n+r)!$ is to be interpreted as $\Gamma(n+r+1)$ if n is not an integer. Put

$$(\tfrac{1}{2}z)^2 = x. \quad \text{..................................(2)}$$

Then

$$\begin{aligned} I_n(2x^{\frac{1}{2}}) &= x^{-\frac{1}{2}n} \sum_{r=0}^{\infty} \frac{x^{n+r}}{r!\,(n+r)!} \\ &= x^{-\frac{1}{2}n} \sum_{r=0}^{\infty} \frac{p^{-(n+r)}}{r!} \\ &= x^{-\frac{1}{2}n} p^{-n} \exp p^{-1}, \quad \text{.....................(3)} \end{aligned}$$

where p denotes d/dx. Hence

$$p^{-n} \exp p^{-1} = x^{\frac{1}{2}n} I_n(2x^{\frac{1}{2}}). \quad \text{.....................(4)}$$

Differentiating and substituting for x we obtain the familiar recurrence relation

$$\frac{d}{dz}\{z^n I_n(z)\} = z^n I_{n-1}(z). \quad \text{.....................(5)}$$

The expression on the left of (4) is equivalent to the integral

$$\frac{1}{2\pi\iota} \int_L \exp\left(\kappa x + \frac{1}{\kappa}\right) \frac{d\kappa}{\kappa^{n+1}}, \quad \text{.....................(6)}$$

if n is positive; and if the path L is replaced by a loop M, passing around the origin and extending to $-\infty$, the result holds without restriction on n. Hence

$$I_n(z) = (\tfrac{1}{2}z)^{-n} \frac{1}{2\pi\iota} \int_M \exp\left(\frac{\kappa z^2}{4} + \frac{1}{\kappa}\right) \frac{d\kappa}{\kappa^{n+1}}, \quad \text{...........(7)}$$

which is equivalent to Schläfli's form*. Putting also

$$\tfrac{1}{2}\kappa z = \lambda, \quad \text{...................................(8)}$$

we have

$$I_n(z) = \frac{1}{2\pi\iota} \int_M \exp \tfrac{1}{2}z\left(\lambda + \frac{1}{\lambda}\right) \frac{d\lambda}{\lambda^{n+1}}, \quad \text{...............(9)}$$

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* Watson, *Bessel Functions*, 175–6.

provided the real part of z is positive. If

$$\frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) = \mu, \quad \text{.........................(10)}$$

$$\lambda = \mu + (\mu^2 - 1)^{\frac{1}{2}}, \quad \text{.........................(11)}$$

the sign of the root being determined by the fact that λ and μ must approach $+\infty$ together, we find

$$I_n(z) = \frac{1}{2\pi\iota}\int_M \exp(\mu z) \frac{d\mu}{(\mu^2-1)^{\frac{1}{2}}\{\mu + (\mu^2-1)^{\frac{1}{2}}\}^n}, \quad \text{......(12)}$$

giving, in operational form,

$$I_n(x) = \frac{p}{\{p + (p^2-1)^{\frac{1}{2}}\}^n (p^2-1)^{\frac{1}{2}}}. \quad \text{...............(13)}$$

This can easily be verified for $n = 0$ by expanding in negative powers of p.

8.2. The integrand in 8.1 (12) has branch points at ± 1, but no other singularities in the finite part of the plane. The path can therefore be modified to two loops from $-\infty$, each passing around one of these branch points. Consider first $I_0(z)$. The loop around $+1$ gives a contribution denoted, in Heaviside's notation*, by $\frac{1}{2}H_0(z)$. We have

$$H_0(z) = \frac{1}{\pi\iota}\int_{+1} \exp(\mu z) \frac{d\mu}{(\mu^2-1)^{\frac{1}{2}}}, \quad \text{...............(1)}$$

and if z is great the exponential is appreciable only when $R(\mu)$ is near unity. Then we can put

$$\mu = 1 - \nu, \quad \text{...............................(2)}$$

and write

$$H_0(z) = -\frac{1}{\pi\iota} e^z \int e^{-\nu z} \frac{d\nu}{(-2\nu + \nu^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{\pi\iota} e^z \int \frac{e^{-\nu z}}{(-2\nu)^{\frac{1}{2}}} \left[1 + \tfrac{1}{2} . \tfrac{1}{2}\nu + \frac{\frac{1}{2} . \frac{3}{2}}{2!}(\tfrac{1}{2}\nu)^2 + \dots\right] d\nu, \quad \text{......(3)}$$

where, if the loop be supposed indefinitely narrow, we must take $(-2\nu)^{\frac{1}{2}}$ positive imaginary on the upper side of the path, and negative imaginary on the lower. Hence

$$H_0(z) = \frac{2}{\pi\iota} e^z \int_0^\infty \frac{e^{-\nu z}}{(2\nu)^{\frac{1}{2}}} \left[1 + \tfrac{1}{4}\nu + \frac{1 . 3}{2!}(\tfrac{1}{4}\nu)^2 + \dots\right] d\nu$$

$$= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^z \left[1 + \frac{1}{8z} + \frac{1^2 . 3^2}{2!}\left(\frac{1}{8z}\right)^2 + \dots\right]. \quad \text{............(4)}$$

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* *Electromagnetic Theory*, 2, 453.

Similarly we may consider a loop around -1 and define

$$K_0(z) = \frac{1}{\pi\iota}\int_{-1} \exp(\mu z)\,\frac{d\mu}{(1-\mu^2)^{\frac{1}{2}}}, \qquad \text{.............}(5)$$

the positive sign being taken for the root when $-1<\mu<+1$. Then we put

$$\mu = -1-\nu, \qquad \text{...........................}(6)$$

and obtain

$$K_0(z) = \frac{2}{\pi}\,e^{-z}\int_0^\infty \frac{e^{-\nu z}}{(2\nu)^{\frac{1}{2}}}\left\{1-\tfrac{1}{4}\nu+\frac{1\,.\,3}{2!}\,(\tfrac{1}{4}\nu)^2-\dots\right\}d\nu$$

$$= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-z}\left\{1-\frac{1}{8z}+\frac{1^2\,.\,3^2}{2!}\left(\frac{1}{8z}\right)^2-\dots\right\}. \qquad \text{.........}(7)$$

By substituting the loop integrals for H_0 and K_0 in the differential equation for I_0, namely,

$$\frac{d^2I_0}{dz^2}+\frac{1}{z}\frac{dI_0}{dz}-I_0=0, \qquad \text{.......................}(8)$$

we find easily that they both satisfy it and therefore constitute a pair of independent solutions. The expansions (4) and (7) are asymptotic, but a fuller discussion is needed before a limit can be assigned to the error involved in stopping the expansions at a given term. Their physical interest is that if the variable z is $\sigma\varpi/c$, where ϖ has its usual meaning as a cylindrical coordinate and σ denotes $\partial/\partial t$, K_0 contains $e^{-\sigma\varpi/c}$ as a factor, and its interpretation by Bromwich's rule will have $\exp\kappa\,(t-\varpi/c)$ as a factor in the integrand, where the real part of κ is positive. It therefore represents a diverging wave. Similarly H_0 represents a pure converging wave. They are also intimately connected with the two Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$. If arg z increases till $z=\iota y$, where y is real and positive, the loops must be swung round till they pass to $+\iota\infty$ instead of $-\infty$. Then

$$H_0(z) = H_0(\iota y) = \left(\frac{2}{\pi y}\right)^{\frac{1}{2}}\exp\iota\,(y-\tfrac{1}{4}\pi)\left(1+\frac{1}{8\iota y}+\dots\right)$$

$$= H_0^{(1)}(y), \qquad \text{....................................}(9)$$

$$K_0(z) = K_0(\iota y) = \left(\frac{2}{\pi y}\right)^{\frac{1}{2}}\exp-\iota\,(y+\tfrac{1}{4}\pi)\left(1-\frac{1}{8\iota y}+\dots\right)$$

$$= -\iota H_0^{(2)}(y). \qquad \text{...................................}(10)$$

To express I_0 in terms of H_0 and K_0, we suppose that the loop around $+1$ passes to $-\infty$ on the upper side of -1. The integral around this loop is $\frac{1}{2}H_0$ by definition. Taking the other loop next, we see that

$(\mu^2-1)^{\frac{1}{2}}$ is negative imaginary between the branch points, and the integral is

$$\frac{1}{2\pi\iota}\int_{-1}\exp(\mu z)\frac{d\mu}{(-\iota)(1-\mu^2)^{\frac{1}{2}}}=-\frac{1}{2\iota}K_0(z). \quad \text{.........(11)}$$

Thus

$$I_0(z)=\tfrac{1}{2}[H_0(z)+\iota K_0(z)]. \quad \text{..................(12)}$$

Comparing with (4) and (7) we seem to have the anomaly of a purely real function being equal to the sum of a purely real one and a purely imaginary one. The explanation is that while I_0 and K_0 are purely real by definition when z is real and positive, we have had to define H_0 by means of a loop passing -1 on the positive imaginary side. When $\mu+1$ is imaginary $(\mu^2-1)^{\frac{1}{2}}$ has a real part, and therefore H_0 is not purely real. If we had defined H_0 by means of a loop passing -1 on the under side we should have had to reverse the sign of ι in the equation

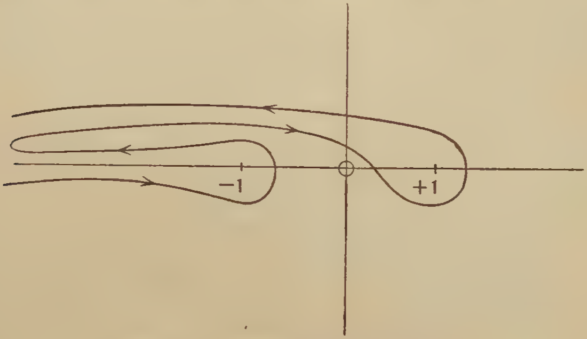

Fig. 8. Contours for H_0 and K_0.

$$H_0(z)=2I_0(z)-\iota K_0(z). \quad \text{.....................(13)}$$

The reason this phenomenon does not show in the asymptotic expansions is that when z is great enough K_0 is smaller than any term in the asymptotic expansion of H_0, and therefore cannot affect this expansion.

The ordinary Bessel function $J_0(z)$ can be defined by

$$J_0(z)=I_0(\iota z), \quad \text{...(14)}$$

$$=\tfrac{1}{2}\{H_0(\iota z)+\iota K_0(\iota z)\}, \quad \text{..............................(15)}$$

$$=\tfrac{1}{2}\{H_0^{(1)}(z)+H_0^{(2)}(z)\}, \quad \text{..............................(16)}$$

$$=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos(z-\tfrac{1}{4}\pi)+\text{higher terms}. \quad \text{...........(17)}$$

In operational form it is given by

$$J_0(x)=\frac{p}{(p^2+1)^{\frac{1}{2}}}. \quad \text{.......................(18)}$$

8.3. In the case of functions of other orders, the extra factor in 8.1 (12) is $\{\mu+(\mu^2-1)^{\frac{1}{2}}\}^{-n}$, which is unity for $\mu=1$ and $e^{-n\pi\iota}$ for $\mu=-1$. Hence we can define an H_n and a K_n, the latter in real form, and the former with a real asymptotic expansion. If this is done the first terms of the expansions will be independent of n, a known result. But when we proceed to the ordinary Bessel function the complex factors affect the argument of the cosine; we have indeed Stokes's approximation

$$J_n(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right). \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

8.4. Heaviside's procedure is to define $H_0(x)$ by the operator

$$\frac{2p}{(p^2-1)^{\frac{1}{2}}}, \text{ and } K_0(x) \text{ by } \frac{2p}{(1-p^2)^{\frac{1}{2}}},$$

in the present notation*. The unsatisfactory feature of these definitions is that whether the fundamental interpretation of an operator is an expansion in powers of $1/p$ or a complex integral, the former operator means $2I_0(x)$ and the latter $\pm 2\iota I_0(x)$. But Heaviside applies a process equivalent to changing the variable from μ to ν as above, and then expanding in ascending powers of this new variable. The resulting operators are equivalent to 8.2 (4) and (7) above; the method really amounts to evaluating the portions of the integrals that arise from the neighbourhoods of the branch points, and therefore gives H_0 and K_0 correctly. But the operators introduced at the start do not represent the functions Heaviside finishes with.

8.5. The present expansions can be applied to the solution of the problem of 6.21. The operational solution was

$$\begin{aligned}\Phi &= -\tfrac{1}{2}\pi\frac{a}{c}\, I_0'\left(\frac{\sigma a}{c}\right) K_0\left(\frac{\sigma\varpi}{c}\right)\\ &= -\tfrac{1}{2}\pi\frac{a}{c}\cdot\tfrac{1}{2}\left(\frac{2c}{\pi\sigma a}\right)^{\frac{1}{2}} e^{\sigma a/c}\left(\frac{2c}{\pi\sigma\varpi}\right)^{\frac{1}{2}} e^{-\sigma\varpi/c}\left(1-\frac{3c}{8\sigma a}\right)\\ &= -\tfrac{1}{2}\left(\frac{a}{\varpi}\right)^{\frac{1}{2}} e^{-\sigma(\varpi-a)/c}\left(\frac{1}{\sigma}-\frac{3c}{8\sigma^2 a}\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\end{aligned}$$

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* $K_0(x)$, as here defined, in accordance with Heaviside's practice, differs from the definition in 6.21, which is that adopted by Watson, following Macdonald; Watson's K_0 is $\frac{1}{2}\pi$ times Heaviside's. Watson comments on the fact that the extra factor obscures the relation between K_0 and the Hankel functions, but hesitates to remove it because the function is already tabulated. But perhaps tables are of less importance than analytical convenience; when Bessel functions occur in problems they more often than not seem to disappear from the final answer.

and the pressure is

$$P = \tfrac{1}{2}\rho\left(\frac{a}{\varpi}\right)^{\frac{1}{2}} e^{-\sigma(\varpi-a)/c}\left(1-\frac{3c}{8\sigma a}\right), \quad \ldots\ldots\ldots\ldots\ldots(2)$$

which is zero till time $(\varpi - a)/c$, then jumps to $\tfrac{1}{2}\rho\,(a/\varpi)^{\frac{1}{2}}$, and proceeds to fall by $\tfrac{3}{8}c/a$ of itself per unit time.

8.6. In much of Heaviside's work use is made of what he calls the generalized exponential function, defined by the series

$$e^x = \sum_{r=-\infty}^{\infty} \frac{x^{n+r}}{(n+r)!} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where n is a proper fraction. The series formally satisfies the differential equation

$$\frac{dy}{dx} = y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and reduces to the ordinary exponential series when n is an integer. Its mathematical peculiarity is that the part of the series corresponding to negative values of n is always divergent. As usual $(n+r)!$ is to be interpreted as $\Gamma(n+r+1)$. This series was recently found by Mr A. E. Ingham and myself to possess an asymptotic property. Suppose we start with a given term, say $r = -m$, where m is a positive integer, and consider the series

$$\begin{aligned} S &= \sum_{r=0}^{\infty} \frac{x^{n-m+r}}{(n-m+r)!} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3) \\ &= \sum_{r=0}^{\infty} p^{-(n-m+r)} \\ &= \frac{1}{2\pi\iota}\int_L \sum_{r=0} \frac{e^{\kappa x}}{\kappa^{n-m+r+1}}\,d\kappa \\ &= \frac{1}{2\pi\iota}\int_L \frac{e^{\kappa x}}{\kappa^{n-m}}\,\frac{d\kappa}{(\kappa-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4) \end{aligned}$$

if L is so chosen that $|\kappa| > 1$ at all points of it. Then the integrand has a pole at $\kappa = 1$, which makes a contribution e^x to the integral. The remainder is equal to

$$S - e^x = \frac{1}{2\pi\iota}\int \frac{e^{\kappa x}\,d\kappa}{\kappa^{n-m}(\kappa-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

taken around a loop surrounding the origin and passing to $-\infty$ at both ends. This is equivalent to

$$\frac{1}{\pi}\sin(m-n)\pi\int_0^\infty e^{-\mu x}\,\frac{\mu^{m-n}}{\mu+1}\,d\mu \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and the integral is less than

$$\int_0^\infty e^{-\mu x}\,\mu^{m-n}\,d\mu = x^{-(m-n+1)}\,\Gamma(m-n+1)$$

$$= \frac{x^{n-m-1}\,\pi}{\Gamma(n-m)\sin(n-m)\pi}.$$

Hence $$|S - e^x| < \frac{x^{n-m-1}}{(n-m-1)!}$$

so that the error caused by starting at a definite term is less than the preceding term, and decreases indefinitely when x becomes great.

8.7. As has been indicated, Heaviside's work is largely concerned with the use of asymptotic series, which he justified mainly by appeal to actual computation. Less defence of such series is now necessary; pure mathematicians have proved their validity in many cases, and even used them themselves. Some of Heaviside's series, however, are in their actual application convergent. Thus, if we consider the asymptotic expansion of $K_0(x)$, namely

$$K_0(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-x} x^{-\frac{1}{2}} \left(1 - \frac{1}{8x} + \frac{1^2 . 3^2}{2!\,(8x)^2} - \dots \right.$$
$$\left. \dots + (-1)^n \frac{1^2 . 3^2 \dots (2n-1)^2}{n!\,(8x)^n} + \dots \right),$$

this is formally divergent as it stands; but in its application to problems of cylindrical waves the argument is $\sigma\varpi/c$, and the general term (allowing for the factor $x^{-\frac{1}{2}}$) is

$$u_n = (-1)^n \frac{1^2 . 3^2 \dots (2n-1)^2}{n\,!\,(8\varpi/c)^{n+\frac{1}{2}}\,\sigma^{n+\frac{1}{2}}} = (-1)^n \frac{1^2 . 3^2 \dots (2n-1)^2\, t^{n+\frac{1}{2}}}{n!\,(8\varpi/c)^{n+\frac{1}{2}}(n+\frac{1}{2})(n-\frac{1}{2})\dots\frac{1}{2}\sqrt{\pi}}$$

$$= (-1)^n \frac{1}{\sqrt{\pi}} \frac{1 . 3 \dots 2n-1}{n!\,(2n+1)\,2^n} \left(\frac{ct}{2\varpi}\right)^{n+\frac{1}{2}}.$$

The general term in the interpretation of $K_0(\sigma\varpi/c)$ is therefore

$$(-1)^n \frac{\sqrt{2}}{\pi} \frac{1 . 3 \dots 2n-1}{n!\,(2n+1)2^n} \left(\frac{ct-\varpi}{2\varpi}\right)^{n+\frac{1}{2}}$$

and the series converges like a binomial expansion provided $ct < 3\varpi$. Though a divergent series is used in the course of the work, it leads to a convergent answer.

8.8. But the operational method raises numerous points concerning the relation between Pure Mathematics and Mathematical Physics in general. Chapter I of this work suffices to show that the operational method gives the correct results when the system considered has a finite number of degrees of freedom; but as soon as we begin to consider continuous systems we find that the solution involves operators not definable in terms of definite integrations, and the method of Chapter I is no longer available for a justification. Bromwich's introduction of complex integration serves three important purposes: it enables us to prove in large classes of cases that the answers obtained to problems of continuous systems are correct; it provides a formal rule for interpreting operators in general; and it gives the answer, in cases where it involves new operators, in a form convenient for direct evaluation by contour integration. It seems curious that physics, dealing entirely with real variables, should find it necessary to use the complex variable to solve its problems in the most convenient way. The use of conformal representation in two-dimensional electrostatics and hydrodynamics is of course exceptional, and arises from the fact that the general real solution of Laplace's equation in two dimensions is the real part of a function of $x + \iota y$; for other equations there is no such explanation. The real reason is that the solutions of the linear differential equations of physics are expressible as linear combinations of analytic functions. In the method introduced by Bromwich himself, for instance, κ is written for $\partial/\partial t$ before solution, and then for the general value of κ the solution is found that satisfies the differential equation and the terminal conditions. Such a combination of these solutions as will satisfy the conditions when $t = 0$ is then constructed in the form of an integral with regard to κ; this integral is the same as would have been obtained by solving operationally and interpreting according to Bromwich's rule. The utility of the complex variable then rests on the fact that analytic functions of it satisfy Cauchy's theorem. I do not think, however, that the complex integral should be regarded as the definition of the operator. In 2.1 (2) the $\phi(p)$ is the fundamental notion, and the integral merely expresses a convenient rule for evaluating it. The rules of Chapter I in my opinion come nearer to the ultimate meaning of the operators, but their extension to the operators that arise in the discussion of continuous systems awaits further investigation.

NOTE

ON THE NOTATION FOR THE ERROR FUNCTION OR PROBABILITY INTEGRAL

The notation given on p. 26 is one that I adopted in 1916 under the impression that it was in general use, and I have since used it in several publications*. My definition was recently queried by a correspondent, and I have not succeeded in tracing its origin. I have, on the other hand, discovered a surprising confusion of other notations. The earliest, due to Gauss†, is

$$\Theta(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}\,dz.$$

No name is given to the function by Gauss, and there is no sign that he meant the notation to be permanent. Of modern writers, Carslaw, Brunt, and Coolidge use this notation. Fourier‡ gives

$$\psi R=\frac{1}{\sqrt{\pi}}\int_R^\infty e^{-r^2}\,dr,$$

but has, so far as I have traced, no modern followers. Jahnke and Emde, in their tables, use

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx,$$

and call this the Fehlerintegral. The same notation is widely used by other German writers. Whittaker and Robinson§ also use it and call the function the Error Function. The notation

$$\operatorname{Erf} x=\int_x^\infty e^{-x^2}\,dx$$

was introduced by J. W. L. Glaisher‖, who also used¶

$$\operatorname{Erfc} x=\int_0^x e^{-x^2}\,dx=\tfrac{1}{2}\sqrt{\pi}-\operatorname{Erf} x.$$

The latter function is also called erf x by R. Pendlebury**.

---

* *Phil. Mag.* 32, 1916, 579–585; 35, 1918, 273; 38, 1919, 718. *M.N.R.A.S.* 77, 1916, 95–97. *Proc. Roy. Soc.* A, 100, 1921, 125–6. *The Earth*, 1924.

† *Werke*, 4, 9. First published 1821.

‡ *Théorie Analytique de la Chaleur*, 1822, 458.

§ *Calculus of Observations*, 1924, 179.

‖ *Phil. Mag.* (4), 42, 1871, 294–302.

¶ *Loc. cit.* 421–436.

** *Loc. cit.* 437–440.

Whittaker and Watson[*] use Glaisher's notation with the meanings of Erf and Erfc interchanged, while Jeans's[†] erf x is $\frac{2}{\sqrt{\pi}}$ times Glaisher's Erf. Numerous other writers use the integrals, but omit to give any special symbol or use only the non-committal "I."

It can hardly be denied, in view of the wide application of the error function to thermal conduction, the theory of errors, statistics, and the dynamical theory of gases, that it merits a distinctive notation. It is equally clear that none of those yet used has obtained general acceptance. Of them, it seems that those involving only the single letters Θ, Φ, Ψ are wholly undesirable. In dynamical problems these letters are in continual use for couples, while Φ is often wanted for a velocity potential and Ψ or ψ for a stream function; ψ is of course also often needed for an angle. Further, Θ has an established meaning already in the theory of elliptic functions, namely, the Theta Function of Jacobi, while Φ has another meaning in the theory of numbers. To avoid confusion in some of the applications of the function it seems necessary to use a combination of letters, and the only one in frequent use is Erf.

As to what function should be denoted by this symbol, the most convenient is indicated by the practice of the compilers of tables, such as Jahnke and Emde and Dale, who tabulate $\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$. This is an odd function and becomes unity when $x=\infty$, two properties that make for analytical convenience and are connected with the fact that this form, including the factor $2\pi^{-\frac{1}{2}}$, usually occurs as such in the solutions of the relevant problems. Accordingly I think that, of the various notations proposed,

$$\operatorname{Erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$$

is the most convenient, and is worthy of wider adoption, though I seem myself to have adopted it under a misapprehension originally.

---

[*] *Modern Analysis*, 1915, 335.

[†] *Dynamical Theory of Gases*, 1921, 34.

INTERPRETATIONS OF THE PRINCIPAL OPERATORS

$$\sigma^{-n} H(t) = 0 \qquad t < 0$$
$$= \frac{t^n}{n!} \quad t > 0.$$

$$\frac{\sigma}{\sigma - a} = e^{at}; \quad \frac{\sigma}{(\sigma - a)^n} = \frac{t^{n-1}}{(n-1)!} e^{at}; \quad \frac{a}{\sigma - a} = e^{at} - 1.$$

$$\frac{f(\sigma)}{F(\sigma)} = \frac{f(0)}{F(0)} + \sum_{a} \frac{f(a)}{aF'(a)} e^{at}.$$

$$\frac{1}{\sigma - a} S(t) = e^{at} \int_0^t S(t)\, e^{-at}\, dt.$$

$$\frac{n\sigma}{\sigma^2 + n^2} = \sin nt; \quad \frac{\sigma^2}{\sigma^2 + n^2} = \cos nt; \quad \frac{n\sigma}{\sigma^2 - n^2} = \sinh nt; \quad \frac{\sigma^2}{\sigma^2 - n^2} = \cosh nt.$$

$$e^{-\sigma x} f(t) = f(t - x).$$

$$\sigma^{\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}; \quad \sigma^{\frac{3}{2}} = -\frac{1}{2\sqrt{\pi}} t^{-\frac{3}{2}}; \quad \sigma^{-\frac{1}{2}} = 2\sqrt{\frac{t}{\pi}}.$$

If $\sigma = h^2 q^2$,

$$e^{-qx} = 1 - \operatorname{Erf} \frac{x}{2ht^{\frac{1}{2}}}; \quad qe^{-qx} = \frac{1}{h(\pi t)^{\frac{1}{2}}} e^{-x^2/4h^2t};$$

$$\frac{1}{q} e^{-qx} = 2h \left(\frac{t}{\pi}\right)^{\frac{1}{2}} e^{-x^2/4h^2t} - x\left(1 - \operatorname{Erf} \frac{x}{2ht^{\frac{1}{2}}}\right);$$

$$\frac{ae^{-qx}}{q + a} = 1 - \operatorname{Erf} \frac{x}{2ht^{\frac{1}{2}}} - \exp(a^2h^2t + ax)\left\{1 - \operatorname{Erf}\left(\frac{x}{2ht^{\frac{1}{2}}} + aht^{\frac{1}{2}}\right)\right\}.$$

$$\frac{p}{\{p + (p^2 - 1)^{\frac{1}{2}}\}^n (p^2 - 1)^{\frac{1}{2}}} = I_n(x); \quad \frac{p}{\{p + (p^2 + 1)^{\frac{1}{2}}\}^n (p^2 + 1)^{\frac{1}{2}}} = J_n(x).$$

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$$\frac{1}{Z(p)} = A(x)$$

is interpreted as the solution of the integral equation

$$\frac{1}{\kappa Z(\kappa)} = \int_0^\infty A(x)\, e^{-\kappa x}\, dx.$$

This equation is then solved by known rules. Numerous electrical applications are given.

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$$\mathop{\mathrm{Lim}}_{l\to 0}\left\{\left(1+\frac{\sigma^2 l^2}{4c^2}\right)^{\frac{1}{2}}+\frac{\sigma l}{2c}\right\}^{-2x/l},$$

when l tends to zero through such values as make x/l integral. This gives the rule 1.8 (5) in terms of definite integration.

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